

### Question 1: 100 total points.

A. *Optimal Portfolio (25 points)*: Consider the following Bellman Equation:

$$\text{For all } k > 0 : V(k) = \max\left\{\frac{(k-s)^{1-\sigma}}{1-\sigma} + \beta E[V(k')]\right\},$$

where  $k' = s\tilde{R}_p$ ,  $\sigma > 0$ ,  $\sigma \neq 1$ ,  $0 < \beta < 1$ ,  $\tilde{R}_p = \omega\tilde{R} + (1-\omega)R$  is constrained to be strictly positive,  $\tilde{R}$  is a strictly positive random return, and  $R$  is a strictly positive risk-free return.

- (a) Derive a closed-form solution for the value function  $V(k)$ . (10 points)
- (b) Determine the policy rules,  $s(k)$  and  $\omega(k)$ . (5 points)
- (c) Determine the condition on  $R$  and  $\tilde{R}$  under which it is optimal to invest a strictly positive amount  $\omega$  in the risky asset. (10 points)

B. *Growth (40 points)*: Consider an economy where a single homogeneous good is traded. the good can be used for consumption or for investment. The economy is populated by a large number of identical agents. Life-time utility of a representative agent is given by

$$E_0 \sum_{t=0}^{\infty} \beta^t [\ln C_t + \gamma(1 - N_t)], \quad 0 < \beta < 1$$

where  $N_t$  is the time spent working. Output is produced according to  $Y_t = A_t K_t^\alpha N_t^{1-\alpha}$ ,  $0 < \alpha < 1$ .  $A_t$  is a stochastic productivity shock. Assume that

$$\ln A_t = \rho \ln A_{t-1} + \varepsilon_t, \quad \varepsilon_t \sim iid(0, \sigma_\varepsilon^2).$$

Further assume that  $\varepsilon_t$  is observed before any period- $t$  decisions are undertaken. Physical capital depreciates at rate  $\delta$ .

- (a) What is the Bellman's equation that corresponds to the above problem. What are the states and controls? (5 points)
- (b) Derive the Euler equation and the FOC for the fraction of time spent. (5 points)
- (c) One of the 'stylized facts' over the last hundred years in the United States is that while GDP per capita has been growing at a constant rate, fraction of life-time spent working has been decreasing. Now assume that  $\delta = 1$ . Is there any hope for this model to match the above 'fact'? In answering this question you could alter the manner in which productivity evolves. Explain and account for it clearly. (20 points)
- (d) Now, assume that  $\gamma = 0$  and  $A_t = A$  for all  $t$ . If  $\delta < 1$ , can the value function take the form  $V(K) = E + F \log(K)$  for constants  $E$  and  $F$ ? Why? (10 points)

- C. *Capital Adjustment* (35 points): Consider the following problem of a firm with convex investment costs. Specifically, the total cost of investment is a function  $c : R \rightarrow R_+$  of gross investment that is strictly increasing, strictly convex and differentiable with  $c(0) = 0$ . The firm operates a technology  $z = f(k)$ , where physical capital  $k \geq 0$  is the single input and  $z \geq 0$  is output. Assume that  $f$  is continuously differentiable, strictly increasing and strictly concave and that

$$f(0) = 0, \lim_{k \rightarrow 0} f'(k) = +\infty, \lim_{k \rightarrow \infty} f'(k) = 0$$

Assume that  $p$  is the price of output,  $q$  the price of capital, and  $r$  the interest rate, with  $\beta = (1 + r)^{-1} \in (0, 1)$ . Assume that capital must be purchased one period in advance and depreciates at the rate  $\delta \in (0, 1)$ . Then the firm's problem is

$$\sup \sum_{t=0}^{\infty} \beta^t \{p f(k_t) - c[k_{t+1} - (1 - \delta)k_t]\}$$

where  $c[\cdot]$  denotes the cost of gross investment (note:  $c[\cdot]$  is a function, not a constant).

- Formulate the functional equation for this problem and show that there is a unique value function  $v^*$  that solves the functional equation. Also show that  $v^*$  is the supremum function for the sequence problem. Be careful to justify each step in your proof with reference to results discussed in class and to recall why these results are applicable.
- Show that  $v^*$  is strictly increasing and strictly concave.
- Show that  $v^*$  is differentiable. Then use the Euler equation to show that the policy function is strictly increasing in current capital.

## Question 2: 100 total points.

- A. (50 points) Consider a version of the Lucas asset pricing model with utility function:

$$u(c) = \log c.$$

Let there be one “tree” and let the aggregate dividend  $d_t$  follow a Markov process with transition function  $F(d'|d)$ . Suppose there is a government that purchases an amount  $g_t$  at date  $t$ , where  $g_t$  does not affect utility, and  $g_t$  follows an exogenous process. In particular,  $g_t = d_t \varepsilon_t$  where  $\varepsilon_t$  is independent and identically distributed over time and  $0 < \varepsilon_t < 1$ . Assume that there are complete markets and we refer to the one tree as “equity.” Government bonds can be issued and retired.

- Assume that all government expenditures are financed by lump-sum taxes per capita in each period  $t$  of  $T_t$ , such that the government balances its intertemporal budget constraint. This tax is independent of all agent characteristics such as wealth and income. Define a competitive equilibrium in this environment.
  - Calculate the price of equity in the economy as a function of the discount rate and expectations of future government spending shocks  $\varepsilon_t$ .
  - Now assume that there is instead an income tax on dividends. At time  $t$ , dividends are taxed at rate  $\tau_t/d_t$  so that  $\tau_t$  units of time  $t$  goods are collected on dividends in  $t$ . Assume a balanced budget rule  $g_t = \tau_t$  and calculate the equilibrium price of equity. Show whether the price of equity is higher or lower than in part (b) and give an economic explanation for your finding.
- B. (50 points) This problem considers the qualitative effects of a growth slowdown. Consider an economy where the population  $N_t$  grows at rate  $n$ , and labor-augmenting technology  $A_t$  grows at rate  $g$ . Household preferences in terms of consumption per capita  $C_t$  are:

$$\sum_{t=0}^{\infty} [\beta(1+n)]^t \frac{C_t^{1-\gamma}}{1-\gamma}.$$

The capital evolution equation is:

$$K_{t+1} = (1-\delta)K_t + F(K_t, A_t N_t) - C_t N_t,$$

where as usual we assume:

$$F(K, N) = K^\alpha N^{1-\alpha}.$$

Suppose the economy is on the balanced growth path, and then there is a fall in the rate of technological change  $g$ .

- What happens to the steady state levels of capital and consumption after the change? Interpret your results.
- What happens to consumption at the time of the change? Interpret your results.
- For a marginal change in  $g$ , find an expression showing how the fraction of output saved on the balanced growth path changes. Does savings increase or decrease?

### Question 3: Monetary Policy (100 points)

This question is based on comparisons of two structural models of output  $y_t$  and inflation  $\pi_t$ : a new-Keynesian model (NK)

$$y_t = \alpha y_{t-1} + \beta \pi_{t-1} + \varepsilon_t$$

and a neoclassical model (NC)

$$y_t = \gamma y_{t-1} + \delta (\pi_{t-1} - E_{t-2} \pi_{t-1}) + \varepsilon_t$$

In these models,  $\varepsilon_t$  is a 0 mean second order stationary process. Assume that the central bank can control the inflation rate  $\pi_t$ , i.e. that the bank can choose the inflation rate each period. For this question, assume that inflation rate rules are based on the joint history of output and inflation, i.e. do not introduce information available to the central bank outside the histories of these series in considering rules.

- A. (15 points). Assume that  $\alpha = \gamma = 1$ . If  $\pi_t = 0 \forall t$  explain how  $y_t$  can be second order stationary for the NK model. Does the same argument apply to the NC model? Explain.
- B. (35 points) Assume that  $\alpha = 1$  and that if  $\pi_t = 0 \forall t$ ,  $y_t$  contains a unit root. Identify an inflation rate rule under (NK) that renders  $y_t$  second order stationary. Does identification of such a rule require knowledge of the Wold representation of  $\varepsilon_t$ ? Explain.

- C. (35 points) Assume that  $\gamma=1$  and that if  $\pi_t=0 \forall t$ ,  $y_t$  contains a unit root. Does there exist an inflation rule under (NC) that can render  $y_t$  second order stationary? Explain.
- D. (15 points). What is the value of the Wold representation of  $\varepsilon_t$  in the setting of optimal policy rules for the (NK) and (NC) models?