1. Dipwit Dull is a financial consultant for Alfred Portzreebie who is retiring from his long academic career. Al knows nothing about risk management. There are two assets, 1 and 2, which yield returns $R_i$, $i = 1, 2$. Al has one unit of wealth and can place fraction $x_i$, $0 \leq x_i \leq 1$, into asset $i$ where $\sum x_i = 1$. Notice that Al is required to choose $x_i$ between 0 and 1, i.e., he is not allowed to “borrow” or “short sell.”

Al’s portfolio return is $R = \sum x_i R_i$. Let $\mu_i$, $\sigma_i^2$, $\sigma_{ij}$ be mean return on $i$, variance of return on $i$, covariance of returns on $i$ with returns on $j$, correlation coefficient, and standard deviations of returns on $i$, $j$. Let $\mu_1 < \mu_2$, $\sigma_1^2 = \sigma_2^2 > 0$, and $\sigma_{12} > 0$. Al likes mean but hates variance. Al’s utility for $R$ is given by $E(R) - (a/2)\text{Var}(R)$, where $E(R)$ denotes the mean of $R$, $\text{Var}(R)$ denotes the variance of $R$, and $a > 0$ measures Al’s dislike of risk.

Dull, being a well-trained economist (having failed the personality requirements for a career in accounting), suggests to Al that, even though $\mu_1 < \mu_2$, he should put some of his money in asset 1 to “diversify” his risk. But Al says to Dull, “Hang on a second. Assets one and two positively covary, so when 1 goes up, so does 2 and vice versa. The variances are the same. Any idiot can see that I should specialize in asset 2 because its mean is higher, so I’m gonna put all my money into asset 2 and fire you!”

Is Al right or is Dull right or is neither right? Both Al and Dull know Al’s utility function. Explain your answer in detail with carefully written proof to back up your answer. I.e., if you believe Al is right, locate sufficient conditions on the parameters of the problem ($\mu_1$, $\mu_2$, $\sigma^2$, $\sigma_{12}$, and $a$, consistent with the assumptions above) so that $x_2 = 1$ is optimal. If you believe Dull is right, locate sufficient conditions such that diversification is optimal. If you believe neither is right, locate sufficient conditions for diversification to be optimal and not to be optimal.

2. Suppose we have two bidders, $A$ and $B$, each of whom may want to purchase up to two units of a good. The seller has exactly two units to sell. Each bidder has diminishing marginal utility, so if she buys two units, the second one purchased does not have as high a value to her as the first. More specifically, let the valuation of bidder $A$ for the $i^{th}$ unit be $v_i^A$ and similarly for $B$. It is common knowledge that $v_2^A = 10$ and $v_2^B = 5$, but $v_1^A$ and $v_1^B$ are private information, known only to the bidder. The beliefs are that these valuations are independently distributed uniformly on $[200, 2000]$.

(a) Suppose the seller uses two second-price auctions to sell the goods. That is, he uses a second-price auction to sell one unit, then the winner receives her unit, and then we have a second-price auction to sell the remaining unit. If $A$ wins the first auction, what happens in the second? If $B$ wins the first auction, what happens in the second?

(b) Given your answer to (a), what is the maximum $A$ would be willing to bid in the first auction? What is the maximum $B$ would be willing to bid? What is the outcome? How much revenue does the seller make from the two auctions?

(c) Suppose instead the seller packages the two units together and uses a second-price auction to sell the package. Specifically, the value of the package to $A$ is $v_1^A + v_2^B$ and analogously for $B$. What is the seller’s expected revenue?
3. Suppose there are two consumption goods with production functions

\[ Q_1 = \left( \sqrt{K_1} + \sqrt{L_1} \right)^2 \]
\[ Q_2 = \left( K_2^{-1} + L_2^{-1} \right)^{-1} \]

where \( K_1 \) is the amount of capital used in the production of good 1, etc.

There are two small countries, A and B, each having the technologies described above. A is endowed with 4 units of K and 50 units of L, and B is endowed with 20 units of K and 3 units of L. Both economies are competitive, and there is free trade in the consumer goods, but the factors of production cannot move between countries. The prices of the consumer goods, established in the world market, are \( p_1 = 1.6 \) and \( p_2 = 9 \). Taking these prices as given, find the competitive equilibrium in each country.

In your equilibrium, are factor prices equal across these two countries? If so, explain why; if not, explain why not.

4. Consider the following two player game.

\[
\begin{array}{cc}
L & R \\
U & 5, 5 & 2, 3 \\
D & 6, 2 & 4, 4 \\
\end{array}
\]

(a) Find all Nash equilibria of the game.

(b) Suppose now that 1 makes his action choice first and that 2 observes this choice before choosing his action. Explain what a strategy is for each player. How many strategies are there in the strategy set for 2? Find all subgame perfect equilibria of this game.

(c) Now suppose 2 can not see the action chosen by 1, but does observe a noisy signal \( \tilde{a} \in \{U, D\} \) before choosing his action. For any choice \( a_1 \in \{U, D\} \) made by 1, \( \tilde{a} = a_1 \) with probability \( 1 - \varepsilon \). (i.e., if 1 selects \( U \), 2's signal, \( \tilde{a} \), equals \( U \) with probability \( 1 - \varepsilon \) and equals \( D \) with probability \( \varepsilon \) and similarly if 1 chooses \( D \)). Sketch the extensive form. Derive all perfect Bayesian equilibria in pure strategies and check how these behave as \( \varepsilon \to 0 \).
(5) Wally World has $N$ customers, each of whom have utility function $u(R, M) = -R^2 + 4R + M$ where $R$ is the number of rides they go on and $M$ is other consumption. They have income of $Y$ and face a price of 1 for $M$. Wally World is a monopoly seller of rides.

(a) Suppose Wally World charges a price of $p$ per ride. What is the optimal choice of $p$ and what are their profits?

(b) Now suppose Wally World can set both a price per ride of $p$ and an entry fee of $E$ to get into Wally World. Given $p$, what is the highest entry fee they can charge? In light of this, what is the optimal $p$ and $E$ for Wally World? How much profit do they earn?

(c) A new group of consumers moves to town, $M$ of them, each with utility function $-R^2 + 4R - F^2 + 4F + (1/2)M$ where $F$ is the amount of food consumed in Wally World. They also each have income $Y$ and face a price of 1 for $M$. Assume that Wally World cannot set different prices for the two different groups of consumers. They can set an entry fee of $E$, a price per ride of $p$, and a price per unit of food of $q$. What is their optimal choice of $E$, $p$, and $q$?

(6) We call a two player game a “signaling game” if it has the following structure:

Stage 1: Nature chooses player 1’s type $t$ from the set $T$ according to probability distribution $p$. Player 1 learns $t$; player 2 does not.

Stage 2: Player 1 chooses a message from the set $M$.

Stage 3: After receiving player 1’s message, player 2 chooses an action from the set $A$. Players then receive their payoffs which may depend on $t$, $m$, and $a$.

The sets $T$, $M$, and $A$ are finite and have at least two elements, and the description of the game is commonly known.

Provide proofs of or counterexamples to each of the following statements:

(i) All Nash equilibria of signaling games are subgame perfect equilibria.

(ii) All subgame perfect equilibria of signaling games are perfect Bayesian equilibria.

(iii) All perfect Bayesian equilibria of signaling games are sequential equilibria.