1. This question is about a public good that can be supplied by only one person, such as becoming Department Chair. The sooner someone steps forward to provide the good, the more time all individuals have to consume the benefits of the good. However, everyone would prefer someone else step forward rather than to do the job herself.

Consider this simple model of this problem. There is an economy of $N$ individuals, $i = 1, ..., N$, that lasts for $T$ periods, $t = 1, ..., T$. Assume that $N < T$. In each period, each agent chooses whether or not to provide the good. Each agent receives one unit of utility for each period the public good is provided (and zero otherwise). If agent $i$ provides the good, this reduces his utility by his cost of provision, $c_i$. However, once the good is provided by at least one person in at least one period, it produces utility for all agents for the rest of the game at zero additional cost. Assume no discounting, so that overall utility for $i$ is just the sum of the payoffs in each period.

Let individual $i$'s cost of providing the public good be $c_i = (3/2)i$. Who will provide the public good and in what period will he provide it? (Hint: If it helps, first think of the problem with $N = 2$, and then generalize.)
Question Two

There are 1000 identical people that live in Frankville. Each consumes two goods $x_1$ and $x_2$ with prices $p_1$ and $p_2$ and has income $m$. The utility function is $u(x_1/A, x_2)$ where $A$ is the average consumption of good one in the town of Frankville. Since each person's own consumption makes such a small part of the average $A$, each takes $A$ as given when consumption decisions on $(x_1, x_2)$ are made.

A. Let the utility function be constant returns Cobb Douglas, i.e. $u(x, y) = x^a y^{1-a}$. Find the demand functions $x_1^*, x_2^*$ for each person in Frankville.

B. Assume Frankville is "small" relative to the rest of the economy so that prices $p_1$ and $p_2$ are determined outside of Frankville. But $A$ is determined within Frankville. Define a rational expectations equilibrium for the determination of $A$ in Frankville by $A = x_1^*$. The Mayor of Frankville proposes a referendum that will impose a tax of $T$ dollars on each unit of good one that is purchased. Proceeds of the tax will be distributed lumpsum and equally to the 1000 citizens of Frankville. Denote this lumpsum transfer by $Tr$ and calculate the equilibrium.

C. Evaluate the utility in the new equilibrium and compare it to the utility in the $T = 0$ equilibrium. If you were living in Frankville would you vote for the Mayor’s referendum? Why or why not? Assume Frankville is located in Wisconsin so the Mayor is honest and all the proceeds of the tax are equally distributed lumpsum to the citizens with no administrative cost.

D. Assume now that Frankville is located in THAT state south of here where the Mayor rips off half of all taxes collected for himself and his cronies. The rest of the proceeds are distributed lumpsum as in part 3 above. Would you still vote for the referendum?
Question 3. Consider a firm that costlessly produces an indivisible good. Each consumer buys at most one unit of the good. There are two types of consumers, high types, who derive utility $h$ from consuming one unit of the good (and zero utility from consuming no units) and low types, who derive utility $l$ from consuming one unit (and zero utility from consuming no units) where $0 < l < h$. The firm cannot observe consumer's types. Proportion $\theta$ of the consumers are high.

A. Suppose that the firm faces a single consumer, and that the firm has a single opportunity to name a price, which the consumer may accept, buying the good, or reject, in which case there is no further interaction between the consumer and firm. Derive the optimal price and expected profits as a function of $\theta$.

B. Now suppose that the firm faces a single consumer, but after the first price is rejected (if it is), then the firm can name a second price, which the consumer can accept or reject. It takes time to name the second price, however, so the utility that a high (or low) consumer obtains when the first price is rejected and second accepted is $Dh$ (or $Dl$), where $0 < D < 1$. The firm's discount factor is also $D$. Derive the optimal prices and expected profits as a function of $\theta$ and $D$. To do so, formulate this problem as a game, identifying carefully the strategies of the players, and derive a perfect Bayesian equilibrium. Compare your answers to those of the previous part, and interpret the results.

C. Repeat your analysis of the second step, but assuming that the firm faces a succession of (individual consumers), with each new consumer arriving only after the previous consumer has left the firm. Explain how it matters whether there are finitely or infinitely many consumers.

D. A firm proudly advertises “In business for 88 years and never a sale.” In light of your answers to the previous questions, why would a firm make such a claim?
4. Consider the following small open economy general equilibrium model. There are two goods denoted X and Y, and two factors of production capital K and labor L. The total endowment of both capital and labor is given and hence in inelastic supply. Denote the price of labor as w and the price of capital as r. Our small open economy faces fixed world prices for both goods. Let Y be the numeraire and denote the relative price of X as p. Technologies are constant returns to scale and production of both X and Y requires both factors of production. For any ratio of factor prices w/r, cost minimizing production of good X uses a greater capital-to-labor ratio than does production of good Y: that is, good X is capital intensive and good Y is labor intensive. The production functions can then be written as:

\[ X = f(K_x, L_x) \quad Y = g(K_y, L_y) \]  

where f and g are constant returns neoclassical production functions with positive but diminishing marginal products. The total endowment of both capital and labor must be allocated across industries, hence:

\[ K_x + K_y = K \quad L_x + L_y = L \]  

This completes the description of the model.

A. Write down the production side equilibrium conditions for this model. These conditions determine factor prices \{w,r\} and outputs \{X and Y\} taking as given world prices and our economy’s endowments.

B. Does this economy produce both goods in equilibrium? What determines whether this economy is diversified (produces both goods) or specialized (produces one good) in production? Give conditions under which both goods will be produced.

C. Suppose our economy is diversified in production and the relative price of X rises in world markets. Find the effect of this small price change on factor prices. If you were a labor owner in this economy has your real income increased or decreased with this change in world prices? If you were a capital owner in this economy has your real income increased or decreased with this change in world prices?

D. Suppose our economy is diversified in production and that our economy’s endowment of labor increases. Find the effect of this small endowment change on outputs X and Y and factor returns w and r. If you were a labor owner in this small open economy how has this increase in labor (perhaps because of immigration) affected your real income?

E. Suppose our economy is diversified in production and our endowments of both capital and labor increase. Let a caret \(^\wedge\) over a variable indicate percentage change; i.e. \(^\wedge\)x = \(\frac{dx}{x}\) and assume the percentage change in capital is greater than that of labor. With given world prices, show that outputs must then change as follows:

\[^\wedge X > ^\wedge K > ^\wedge L > ^\wedge Y\]
Question 5

1. Consider the normal form game in the figure below.
   a. Compute all Nash equilibria of this game.
      Now suppose that this game is played repeatedly, and that the players' common
discount factor is very close to 1.
   b. Graph the set of payoffs which are attainable in subgame perfect equilibrium
      using Nash reversion strategies.
   c. Graph the full set of payoffs attainable in subgame perfect equilibrium.

\[
\begin{array}{ccc}
  & a & b & c \\
A & 3, 2 & 0, 0 & 2, 1 \\
B & 0, 0 & 2, 3 & 0, 1 \\
\end{array}
\]

2. Consider an infinitely repeated game $G^\omega$ based on repeated play of some finite
   normal form game $G$. Players discount payoffs at rate $\delta \in (0, 1)$.
   Are the following statements equivalent? Provide a proof or a counterexample.

   I. Strategy profile $\sigma$ is a Nash equilibrium of $G^\omega$.
   II. For all players $i$ and finite histories $h'$ which occur with positive probability
       under $\sigma$, there is no strategy $\hat{\sigma}_i$ which agrees with $\sigma_i$ at all histories except $h'$ and
       which yields a higher payoff than does $\sigma_i$ starting from history $h'$.

   Hint: You might want to think about the following stage game:

\[
\begin{array}{cc}
  & L & R \\
T & 3, 3 & 3, 3 \\
M & 2, 2 & 2, 2 \\
B & 1, 1 & 1, 1 \\
\end{array}
\]
6. Two Unrelated Questions on Auctions.

A) A seller has two units of a good which two bidders are interested in. The units are complements, so the value of a unit is higher if you already have one. More specifically, bidder A finds only one unit valueless, while both units together are worth 10 to him. Bidder B values one unit at \( v \in (0, 4) \) and values two units at 8. All these facts (including the value of \( v \)) are common knowledge.

If the seller bundles the two units together and has a second price auction to sell the package, which bidder will win and how much money will the seller make?

If instead the seller has a second price auction to sell the first unit followed by another second price auction to sell the second unit, who wins and how much money does the seller make? (Hint: It depends on \( v \).) When does the seller prefer putting the units together and having one auction and when does he prefer having two auctions?

B) A seller has one unit of an indivisible good which has two potential buyers. The valuation of buyer \( i \) for this good is \( v_i \), where \( v_1 \) and \( v_2 \) are independent draws from the distribution function \( F \). The seller uses a first price auction to sell the good. (Yes, this is just the usual independent private values first price auction with two bidders.)

Before the auction, the seller spent money changing the characteristics of the good. More specifically, he invested some amount of money which effectively changed the probability distribution from which the valuations are drawn to \( \tilde{F} \). All parties are risk neutral. These facts (including the exact distributions involved) are all common knowledge. Assume that the expected value of a draw from \( F \) and a draw from \( \tilde{F} \) are the same.

You work at the Federal Trade Commission. Your supervisor is suspicious because the seller's investment does not change the expected value of a draw and suspects something socially inefficient. Is your supervisor correct to be suspicious? If the answer is ambiguous, what would you need to know to determine it?

Hint: For any numbers \( a \) and \( b \), \( \min\{a, b\} + \max\{a, b\} = a + b \).