Question 1

*The Transfer Paradox:* Consider an exchange economy with two agents and two goods. The total resources of the economy are \( w = (10, 20) \) and preferences are represented by the utility functions

\[
  u^1(x, y) = xy^2 \quad \text{and} \quad u^2(x, y) = x^2 y
\]

where \( x \) denotes consumption of good 1 and \( y \) denotes the consumption of good 2.

a) Compute the set of Pareto efficient allocations for this economy and represent it in an Edgeworth box.

b) The *transfer paradox* occurs when an agent—say agent 1—makes a gift to another agent—say agent 2—and the agent who receives the gift is worse off in equilibrium after the gift than before. In other words, in a simple two-agent two-good economy, the transfer paradox occurs if there is a competitive equilibrium of \( \mathcal{E}(u^1, u^2, w^1 - a, w^2 + a) \) with \( a > 0 \), in which agent 2 is worse off than in some competitive equilibrium \( \mathcal{E}(u^1, u^2, w^1, w^2) \). Illustrate in an Edgeworth box that if the transfer paradox occurs the budget lines in the equilibrium of \( \mathcal{E}(u^1, u^2, w^1 - a, w^2 + a) \) and in the equilibrium of \( \mathcal{E}(u^1, u^2, w^1, w^2) \) must cross between the point representing the initial allocations [that is, \((w^1 - a, w^2 + a)\) and \((w^1, w^2)\)] and the Pareto set. (You can take an Edgeworth box and a Pareto set which look similar the ones you derived in question (a).)

c) Deduce from (b) that the transfer paradox can occur only in economies which have multiple equilibria for some values of the initial endowments \((w^1, w^2)\).

d) Compute the competitive equilibrium (equilibria) for all possible distributions of the initial resources \((w^1, w^2)\) in the economy presented above, and show that the transfer paradox cannot occur in this economy.
Question 2

Consider a pure exchange economy with three consumers (labeled 1, 2 and 3) and three goods (labeled x, y and z). Agent i's consumption vector is \( (x_i, y_i, z_i) \). Each agent is endowed with only one type of good: \( \omega_1 = (1,0,0) \), \( \omega_2 = (0,1,0) \) and \( \omega_3 = (0,0,1) \) respectively. The consumers' preferences can be represented by utility functions, as follows:

\[
\begin{align*}
U_1(x_1,y_1,z_1) &= x_1(y_1 + z_1) \\
U_2(x_2,y_2,z_2) &= x_2y_2 \\
U_3(x_3,y_3,z_3) &= x_3z_3.
\end{align*}
\]

a. Find a Walrasian equilibrium for this economy under the condition that good z cannot be traded.
   i. Is this equilibrium Pareto optimal?

b. Assume now that there is a complete set of markets in all three goods. Find a Walrasian equilibrium for this case.
   i. Is this equilibrium Pareto optimal?

c. Can you Pareto-rank the equilibria that you found? Interpret your findings.

d. Consider the following allocation:
   \( \{ (x_1, y_1, z_1) = (5/8, 1/2, 3/4), (x_2, y_2, z_2) = (1/4, 1/2, 0), (x_3, y_3, z_3) = (1/8, 0, 1/4) \} \). Is this in the core?

e. Describe the core of this economy, as completely as you can.
Question 3

A seller has one unit of a good which she may sell to a buyer. The seller has private information about her valuation of the good, \( v \), which is drawn from \([0, 1]\) according to the uniform distribution. When the seller’s valuation of the good is \( v \), the buyer’s valuation is \( kv \), where \( k > 1 \). The buyer does not observe his valuation, however, but does have accurate knowledge of the distribution of the seller’s valuation. Both players are risk neutral.

(a) Suppose that the buyer makes a take-it-or-leave-it offer to the seller. That is, the buyer offers a price at which he is willing to buy, the seller either accepts or rejects, and rejection results in no sale. Describe all subgame perfect equilibria in pure strategies. How does your analysis depend on the value of \( k \)?

(b) Suppose now that the seller makes a take-it-or-leave-it offer. That is, the seller charges a price, the buyer either accepts or rejects, and rejection results in no sale. Describe all perfect Bayesian equilibria in pure strategies. How does your analysis depend on the value of \( k \)?
Question 4

Consider a general equilibrium model of a country that produces and consumes two goods, M and H, where M is manufactures and H is harvest from a renewable resource. Good M is the numeraire with price normalized to one and is produced with labor alone with constant returns, so by choice of units, one unit of labor produces one unit of M. There are N agents in the economy. Harvest H is given by \( H = aS_L \), where \( L \) is aggregate labor allocated to harvesting, \( S \) is the current stock of the renewable resource, \( "a" \) is the level of harvesting technology, and \( M \) is given by \( M = N - L \). Production in both the manufacturing sector and the resource sector is carried out by competitive profit maximizing firms under conditions of free entry. Hence price equals unit cost of production in equilibrium. Hence when stock of the resource is \( S \), then equilibrium price of the resource good is given by

\[ p = \frac{w}{aS}, \]

where \( w \) is the wage, and labor is assumed to be freely mobile. Each agent is endowed with one unit of labor and maximizes \( U(h, m) \) subject to

\[ ph + m = w. \]

Answer the following questions. There are 10 points. Point values for each part are given in parentheses.

I. (2) For a given stock \( S \) of the resource, find temporary equilibrium, i.e. short run equilibrium, for demands generated by,

(3) \[ U(h, m) = h^b m^{1-b} \quad 0 < b < 1 \quad \text{(Cobb Douglas)} \]

(4) \[ U(h, m) = \min\{h, m\} \quad \text{(Leontief)}. \]

Denote your solutions for the short run equilibrium harvest as a function of \( S \) by \( CDH(S), LH(S) \) for the Cobb Douglas and Leontief cases respectively.

II. (4) A long run steady state harvest and stock \((H, S)\) is defined by

(5) \[ H = CDH(S) = G(S), \quad H = LH(S) = G(S) \]

where \( G(S) \) is the natural growth rate of the resource, and \( CDH(S), LH(S) \) are your short run equilibrium harvest schedules for the Cobb Douglas and Leontief cases from above. For

(6) \[ G(S) = rS(1 - S/K), \quad r > 0, \quad K > 0, \]

study existence and uniqueness of long run steady state general equilibrium values for the stock \( S \) for the Cobb Douglas demands and Leontief demands above. Locate sufficient conditions for existence of a positive long run equilibrium value of \( S \). Explain
the economics. How many positive equilibria are there (if any exist at all) for the Cobb Douglas case and for the Leontief case? Explain the economics.

III. (4) The social long run optimum is given by

Maximize

(7) \( NU(h,m) \) subject to,

(8) \( aSL=G(S), M=N-L, h=H/N, m=M/N. \)

Let \( G(S)=rS(1-S/K). \)

(i) For the Leontief case, compare long run equilibrium welfare with social long run optimum welfare. Explain your findings.

(ii) For the Cobb Douglas case, compare long run equilibrium welfare with social long run optimum welfare. Explain your findings.

(iii) Compare your findings in (i) and (ii) above. Explain why a qualitative difference appears, if any exists.
Question 5

A. (2 points)
Let \( \Gamma \) be an extensive form game of perfect information. Zermelo's Theorem tells us that if no player is indifferent between his payoffs at every pair of terminal nodes, then \( \Gamma \) has a unique subgame perfect equilibrium. Is this still true if \( \Gamma \) is allowed to contain moves by Nature? Provide a proof or a counterexample.

B. (3 points)
Let \( G_0 = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}) \) be a zero sum game.

(i) Suppose that \( \sigma_i \in \Delta S_i \) is a rationalizable strategy for player 1. Must \( \sigma_i \) be a maxmin strategy for player 1? Provide a proof or a counterexample.

(ii) Suppose that \( \sigma_i \in \Delta S_i \) is a maxmin strategy for player 1. Must \( \sigma_i \) be a rationalizable strategy for player 1? Provide a proof or a counterexample.

C. (3 points)
Let \( G = (N, \{S_i\}_{i \in N}, \{u_i\}_{i \in N}) \) be a normal form game.

(i) Give a formal definition of a correlated equilibrium of \( G \). If \( \sigma \) is a correlated equilibrium, explain how a planner can use a signaling device to induce rational players to play according to \( \sigma \).

(ii) What condition(s) must \( \sigma \) satisfy to also be a Nash equilibrium of \( G \)? State the conditions formally, and interpret them in the context of the signaling device considered in part (i).

D. (2 points)
Suppose that two players play a normal form game twice, and that the choices made in the first period are observed before the second period occurs. Is it possible in a subgame perfect equilibrium for player 1 to choose a strategy which is dominated in the stage game during at least one of the two periods? Provide an example or a proof to the contrary.
Question 6

A principal must hire an agent. The job to be filled by the agent can be either good (G) or bad (B). The principal does not know the type of the job, knowing only that it is G with probability $\rho \in (0, 1)$. (For example, the agent may be a salesperson, and the product or territory to which the agent is to be assigned might be one in which it is easy to sell (G) or difficult to sell (B).) The agent (only) knows the type of job. Upon being hired, the agent decides whether to exert high effort (H) or low effort (L). The output, as a function of the type of job and the agent’s effort, is given by:

<table>
<thead>
<tr>
<th>Good job, G</th>
<th>Bad job, B</th>
</tr>
</thead>
<tbody>
<tr>
<td>High effort, $H$</td>
<td>4</td>
</tr>
<tr>
<td>Low effort, $L$</td>
<td>2</td>
</tr>
</tbody>
</table>

Low effort is costless to the agent, while high effort imposes a cost of 1. The principal observes the output and then makes a payment to the agent. Payments to the agent cannot be negative. The mapping from outputs to payments is called a contract. The principal maximizes the expected value of output minus payments to the agent. The agent maximizes the expected value of payments minus the cost of effort, and has a reservation utility of 0. Answer the following four questions about this relationship:

- What is the optimal contract for the principal to offer to the agent? Explain how your answer depends on the value of $\rho$ and interpret.

- Now suppose that the relationship between the principal and the agent lasts two periods. The type of job is determined at the beginning of the relationship, and remains the same for both periods. The principal offers the agent a new contract in each period. Hence, the agent learns the type of job. Then the principal offers a period-one contract, the agent chooses an effort level, and outputs and payments are realized. Then the principal offers a second-period contract, the agent chooses an effort level, and output and payments are realized. Assume, in answering this part of the question, that the principal always induces high effort in period one from an agent in a $B$ job, and optimally induces high effort from both types of agent in the single-period contract examined in the previous part. For the two possible remaining configuration of first-period effort levels that the principal might induce ($G$ agent chooses $H$ or $G$ agent chooses $L$), identify and explain the optimal contract that achieves this configuration.

- Suppose now that at the beginning of the two-period relationship, the principal can commit to the period-one payment scheme and also to the (possibly history-dependent) period-two payment scheme. For each of the cases examined in the previous section, explain why this ability to commit can or cannot make the firm better off. Under what conditions will commitment be optimal?

- Some organizations routinely shuffle their employees between jobs. This practice is sometimes criticized because it sacrifices job-specific human capital that the employees may have accumulated. In light of your answers to the previous parts of this question, why this practice might be optimal?