Agency Business Cycle

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Abstract

We propose a novel theory of unemployment fluctuations. Firms need to fire non-performing workers with some positive probability in order to provide them with an ex-ante incentive to exert effort. In order to provide this incentive at the lowest cost, firms load the firing probability on the states of the world where the worker’s share of the gains from trade is high. When there are aggregate decreasing returns to scale to the value of unemployment, the states of the world where the worker’s share of the gains from trade is high are the states where unemployment is high. Hence, an individual firm finds it optimal to fire its non-performing workers exactly at the time when other firms fire their workers. The strategic complementarity between the resolution to the agency problem of different firms leads to endogenous fluctuations in unemployment, job-destruction and job-finding rates.

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1 Introduction

The paper proposes a novel theory of unemployment fluctuations, which are not driven by exogenous fluctuations in labor productivity or other fundamentals, but by a strategic complementarity in the solution of the agency problems facing different pairs of firms and workers. According to our theory, firms need to fire non-performing workers with some positive probability in order to provide them with an ex-ante incentive to exert effort. In order to provide this incentive at the lowest cost, firms load up the firing probability on the states of the world where the worker’s share of the gains from trade is high. When there are aggregate decreasing returns to scale to the value of unemployment, the states of the world where the worker’s share of the gains from trade is high are the states where unemployment is high. Therefore, an individual firm finds it optimal to fire its non-performing workers exactly at the time when other firms fire their workers. The strategic complementarity between the optimal resolution to the agency problem of different firms leads to endogenous fluctuations in unemployment, job-destruction and job-finding rates.

We consider a search-theoretic model of the labor market in the spirit of Mortensen (1971), Pissarides (1985) and Mortensen and Pissarides (1994). Unemployed workers spend time looking for vacant jobs and firms spend resources creating and maintaining vacancies. Unemployed workers and vacant jobs come together through an aggregate matching function. Once matched, workers and firms bargain over the terms of trade and engage in production. The output of production depends stochastically on the effort exerted by the worker, which cannot be directly observed by the firm. Firms and workers bargain over the terms of short-term contracts, which are renegotiated every period.

In the first part of the paper, we characterize the optimal contract between a firm and a worker. We show that, if wages cannot be contingent on the realization of output, the optimal contract prescribes that the firm should fire the worker with some positive probability in case of poor performance. The firing probability prescribed by the optimal contract is just enough to give the worker the incentive to exert effort. Moreover, we show that the optimal contract loads the firing probability on the states of the world where the gains from trade accruing to the worker are high relative to the gains from trade accruing to the firm. This result is intuitive and yet novel. Firings are ex-ante optimal, as they provide workers with the incentive to exert effort, but they are ex-post inefficient, as they destroy valuable relationships which causes losses to both workers and firms. However, the ex-post loss that provides ex-ante incentives is the one hitting the worker. The loss hitting the firm is just “collateral damage.” By loading the firing probability up onto the state of the world where the worker’s gains from trade are relatively high, the optimal contract provides incentives at the lowest cost. Finally, we show that the optimal contract prescribes
a wage such that the product between the firm’s gains from trade and the worker’s marginal utility of consumption is equated to the product between the worker’s gains from trade and the firm’s marginal utility of consumption. This is the standard optimality condition for Nash bargaining. The condition implies that, if workers are risk-averse and firms are risk-neutral, the gains from trade accruing to the worker are high relative to the gains from trade accruing to the firm in the states of the world where the bargained wage is low.

In the second part of the paper, we prove the existence of an equilibrium in which firms load up the probability of firing their non-performing workers on a particular realization of an ostensibly uninformative and payoff irrelevant signal, i.e. a sunspot. The intuition behind this result is simple. Suppose that there is a realization of the sunspot on which firms load the probability of firing their non-performing workers. For that realization of the sunspot, unemployment will be high. If high unemployment is associated with a low value of being unemployed—because of, say, decreasing returns to scale in the aggregate matching function or decreasing returns to scale in non-market production—then the wage of employed workers will be low for that realization of the sunspot. In turn, if the wage of employed workers is low, the share of the gains from trade accruing to the workers will be high relative to the share accruing to the firms and, in light of the properties of the optimal contract, firms will find it optimal to load up the probability of firing their workers exactly on that realization of the sunspot, which verifies our initial conjecture. We formally prove the conjecture in the case of an aggregate matching function with decreasing returns to scale.

In the last part of the paper, we describe the features of an equilibrium with coordinated firings. First, the equilibrium features unemployment fluctuations that are potentially large and yet are uncorrelated with labor productivity. This feature is consistent with the fact that—from 1990 to 2012—the correlation between unemployment and labor productivity in the US has been basically zero. According to our theory, unemployment fluctuations are not caused by shocks to labor productivity or other fundamentals. Instead, unemployment fluctuations emerge endogenously from the strategic complementarity between the firing decisions of different firms: If one firm decides to concentrate the probability of firing its non-performing workers in a particular state of the world, the other firm has an incentive to concentrate its firings on exactly the same state of the world. Second, the equilibrium is such that random increases in the job-destruction rate are driving the unemployment rate up and, in turn, increases in the unemployment rate induce—because of decreasing returns to scale to the matching function—a decline in the job-finding rate. This feature of the equilibrium is consistent with the observation that, in the data, fluctuations in the job-destruction rate leads both fluctuations in the unemployment rate and in the job-finding rate. Third, the equilibrium can naturally generate unemployment fluctuations that, like
in the data, are highly asymmetric, i.e. such that the unemployment rate increases faster in a recession than it declines in a recovery.

The main contribution of the paper is to present a novel theory of labor market fluctuations and, of business cycles in general, which is based on the interaction of the agency problem faced by different firms in the economy. Our agency theory of business cycles stands in sharp contrast with the real business cycle theory, which emphasizes technology shocks as the main driving force of cyclical fluctuations (see, e.g., Kydland and Prescott 1982 in the context of Walrasian markets or Shimer 2005 in the context of a search theoretic model of the labor market). Our agency theory also differs from new-Keynesian business cycle theory, which emphasizes the role of monetary shocks and nominal rigidities as a key driving force of business cycles. Our theory is closest to theories of self-fulfilling fluctuations. In contrast to these theories, though, self-fulfilling fluctuations in our model do not occur because of technological complementarities between firms (e.g. Farmer and Guo 1994, Benhabib and Farmer 1994, Mortensen 1999) or because of demand externalities (e.g. Diamond 1982, Heller 1986, Cooper and John 1988, Kaplan and Menzio 2014), but because of strategic complementarities in the firm’s incentives to fire workers.

Our agency theory of business cycles rests on two key elements. First, firms must fire non-performing workers along the equilibrium path. In this paper, firms fire non-performing workers along the equilibrium path because firing is the only instrument that firms can use to give workers an incentive to exert effort. The current wage cannot be used to provide incentives because it is paid out before the realization of output. Future wages cannot be used to provide incentives because they are renegotiated period by period. These are extreme assumptions, but we make them to better illustrate our theory in the simplest possible way. There are several examples in the literature in which—even with fully history-dependent payments—the principal finds it optimal to terminate a non-performing agent along the equilibrium path (see, e.g., Clementi and Hopenhayn 2006). In these papers, termination happens because output is a stochastic function of effort and agents are protected by some form of limited liability. Second, firms must capture a lower share of the gains of trade when unemployment is higher. In this paper, the share of the gains from trade accruing to firms decreases with unemployment because, when unemployment is higher, the value of unemployment is lower (due to decreasing returns to scale in the matching function), the wage bargained by employed workers is lower and, as long as workers are more risk averse than firms, the share of the gains from trade accruing to workers is higher. Clearly, there are many other assumptions that would lead to the same link between unemployment and the division of the gains from trade between firms and workers.
2 Environment and Definition of Equilibrium

2.1 Environment

Time is discrete and continues forever. The economy is populated by a measure 1 of identical workers and by a positive measure of firms. Every worker has preferences described by \( \beta^t [v(w_t) - ce_t] \), where \( \beta \in (0, 1) \) is a discount factor, \( v(w_t) \) is a twice differentiable, strictly increasing and concave function of period-\( t \) consumption \( w_t \) and \( e_t \in \{0, 1\} \) is an indicator of the worker’s effort on the job in period \( t \) and \( c > 0 \) is the disutility of effort. We assume that the first derivative of the worker’s periodical utility is such that \( v'(w) \in [\underline{w}', \bar{w}'] \) for all \( w \in \mathbb{R} \), with \( \underline{w}' > \bar{w}' > 0 \), and that the second derivative is such that \( -v''(w) \in [\underline{w}'', \bar{w}''] \) for all \( w \in \mathbb{R} \), with \( \underline{w}'' > \bar{w}'' > 0 \). Every firm has preferences described by \( \sum \beta^t [p_t - kv_t] \), where \( p_t \) is the firm’s profit in period \( t \), \( v_t \) is the number of vacancies created by the firm in period \( t \) and \( k > 0 \) is the utility cost of creating a vacancy. Every firm operates a constant return to scale technology that we shall describe below.

Every period \( t \) is divided into five stages: sunspot, matching, separation, bargaining and production. In the first stage, an aggregate shock, \( s_t \), is realized. The aggregate shock is i.i.d. over time and takes the value \( B \) with probability \( \pi_B \in (0, 1) \) and the value \( G \) with probability \( \pi_G = 1 - \pi_B \), where \( B \) is mnemonic for Blue and \( G \) is mnemonic for Green. The aggregate shock has no effect on technology or preferences, but, as we shall see, it may allow coordination among firm-worker pairs. Thus, we refer to \( s_t \) as a sunspot.

In the second stage, firms create vacancies at the unit cost \( k \). Then unemployed workers, \( u_{t-1} \), and vacancies, \( v_t \), come together through a decreasing return to scale matching function \( M(u_{t-1}, v_t) \). We denote as \( \theta_t \) the vacancy-to-unemployment ratio, \( v_t/u_{t-1} \), and we refer to this as the tightness of the labor market. We denote as \( \lambda(\theta_t, u_{t-1}) \) the probability that an unemployed worker meets a vacancy, i.e. \( \lambda(\theta_t, u_{t-1}) = M(u_{t-1}, u_{t-1}\theta_t)/u_{t-1} \). Similarly, we denote as \( \eta(\theta_t, u_{t-1}) \) the probability that a vacancy meets an unemployed worker, i.e. \( \eta(\theta_t, u_{t-1}) = M(u_{t-1}, u_{t-1}\theta_t)u_{t-1}/\theta_t \). We assume that \( \lambda(\theta_t, u_{t-1}) \) is strictly increasing in \( \theta_t \) and strictly decreasing in \( u_{t-1} \) and Moreover, we assume that \( \eta(\theta_t, u_{t-1}) \) is strictly decreasing in \( \theta_t \) and in \( u_{t-1} \).

In the third stage, workers employed at the beginning of the period, \( 1 - u_{t-1} \), are fired by their employer with a probability that depends on the prescriptions of the contract they signed at the bargaining stage of the previous period.

In the fourth stage, new and continuing firm-worker pairs bargain over a contract that regulates the terms of employment until the bargaining stage of next period. Specifically, firm-worker pairs bargain over the wage that should be paid to the worker, \( w_t \), over the effort that the worker should put into production, \( e_t \), and over the probability that the worker
is fired at the next separation stage conditional on the realization of output in the current period and on the realization of the sunspot in the next period, \( \{d_{iH}, d_{iL}\} \) for \( i = \{B, G\} \). Following the tradition of the literature on random search in the labor market, we assume that the outcome of the bargain is given by the Axiomatic Nash Bargaining Solution where the disagreement outcome is given by the value of being unmatched.

In the last stage, production and consumption take place. An unemployed worker home produces and consumes \( b > 0 \) units of output. An employed worker privately chooses an effort level \( e_t \). The output of the firm-worker pair is then determined as a stochastic function of the worker’s private effort. In particular, if the worker exerts effort \( e_t \in \{0, 1\} \), the output of the match is equal to \( y_H \) with probability \( q(e_t) \) and to \( y_L \) with probability \( 1 - q(e_t) \), where \( y_H > y_L \geq 0 \) and \( q(1) > q(0) \) and \( q(1) < 1, q(0) > 0 \). Before output is observed, the worker is paid the wage \( w \) prescribed by his employment contract.

### 2.2 Definition of equilibrium

Let \( u \) denote the unemployment rate at bargaining stage in the current period. Let \( U(u) \) and \( W(u) \) denote the lifetime utility of a worker who is, respectively, unemployed and employed at the beginning of the bargaining stage, and let \( V(u) \) denote the difference \( W(u) - U(u) \). Let \( J(u) \) denote the lifetime profit of a firm from employing a worker at the beginning of the bargaining stage. Let \( \theta_i(u) \) denote the tightness of the labor market at the matching stage of next period given that next period’s sunspot is \( i = \{B, G\} \). Let \( g_i(u) \) denote the unemployment rate at the bargaining stage of next period given that next period’s sunspot is \( i = \{B, G\} \). Finally, let \( x(u) = (w(u), e(u), d_{HI}(u), d_{LI}(u)) \) denote the prescriptions of a contract that the firm and the worker agree upon at the bargaining stage of the current period.

The lifetime utility of an unemployed worker, \( U(u) \), is given by

\[
U(u) = v(b) + \beta \sum_{i = \{B, G\}} \pi_i \{U(g_i(u)) + \lambda(\theta_i(u), u) [W(g_i(u)) - U(g_i(u))]\}
\]  

(2.1)

In the current period, the worker home produces and consumes \( b \) units of output. In the next period, the worker finds a job with probability \( \lambda(\theta_i(u), u) \) and does not find a job with probability \( 1 - \lambda(\theta_i(u), u) \). In the first case, the worker’s continuation utility is \( W(g_i(u)) \). In the second case, the worker’s continuation utility is \( U(g_i(u)) \).
The lifetime utility of an employed worker, $W(u)$, is given by
\[
W(u) = v(w(u)) - e(u)c + \beta \sum_{i=\{B,G\}} \pi_i U(g_i(u)) \\
+ \beta \sum_{i=\{B,G\}} \pi_i [1 - q(e(u))d_iH(u) - (1 - q(e(u)))d_iL(u)] (W(g_i(u)) - U(g_i(u))).
\] (2.2)

In the current period, the worker consumes $w(u)$ units of output and exerts effort $e(u)$. If production is successful in the current period, the worker is fired with probability $d_iH(u)$ and remains employed with probability $1 - d_iH(u)$. Similarly, if production is unsuccessful in the current period, the worker is fired with probability $d_iL(u)$ and remains employed with probability $1 - d_iL(u)$. If the worker is fired, his continuation utility is $U(g_i(u))$. If the worker remains employed, his continuation utility is $W(g_i(u))$.

Let $G(x, u)$ denote the difference between the lifetime utility of a worker employed at an arbitrary contract $x = (w, e, d_iH, d_iL)$ and the lifetime utility of an unemployed worker. Then $G(x, u)$ is given by
\[
G(x, u) = v(w) - v(b) - ec + \beta \sum_{i=\{B,G\}} \pi_i [1 - q(e)d_iH - (1 - q(e))d_iL - \lambda(\theta_i(u), u)] V(g_i(u)).
\] (2.3)

The difference $V(u) = W(u) - U(u)$ can be written as
\[
V(u) = G(x(u), u).
\] (2.4)

Let $F(x, u)$ denote the lifetime profit for a firm employing a worker at an arbitrary contract $x = (w, e, d_iH, d_iL)$. Then, $F(x, u)$ is given by
\[
F(x, u) = y_L + q(e)(y_H - y_L) - w \\
+ \beta \sum_{i=\{B,G\}} \pi_i [1 - q(e)d_iH - (1 - q(e))d_iL] J(g_i(u)).
\] (2.5)

In the current period, the expected output produced by the worker is $y_L + p(e)(y_H - y_L)$ and the wage paid to the worker is $w$. If production is successful in the current period, the firm fires the worker with probability $d_iH$ and retains the worker with probability $1 - d_iH$. Similarly, if production is unsuccessful in the current period, the firm fires the worker with probability $d_iL$ and retains the worker with probability $1 - d_iL$. In the first case, the firm’s continuation profit from employing the worker is zero. In the second case, the firm’s continuation profit from employing the worker is $J(g_i(u))$. The lifetime profit for a firm employing a worker at the optimal contract $x(u)$ is given by
\[
J(u) = F(x(u), u).
\] (2.6)
The optimal contract $x(u)$ is given by the Axiomatic Nash Bargaining Solution. A contract $x$ is feasible if and only if it is incentive compatible, in the sense that it gives the worker the incentive to exert the prescribed level of effort $e$. A contract $x$ is the Axiomatic Nash Bargaining Solution if and only if it maximizes, among all feasible contracts, the gains from trade accruing to the firm, $F(x, u)$, taken to the power of $\gamma$ times the gains from trade accruing to the worker, $G(x, u)$, taken to the power of $1 - \gamma$, where $\gamma \in (0, 1)$ is the firm’s bargaining power. Formally, the contract $x(u)$ solves

$$\max_{x=(w, e, d_H, d_L)} F(x, u)^\gamma G(x, u)^{1-\gamma},$$

subject to the worker’s incentive compatibility constraint

$$c \geq \beta(p_1 - q_0) \sum_{i=\{B,G\}} \pi_i (d_iL - d_iH) V(g_i(u)), \quad \text{if } e = 1,$$

$$c < \beta(p_1 - q_0) \sum_{i=\{B,G\}} \pi_i (d_iL - d_iH) V(g_i(u)), \quad \text{if } e = 0.$$

In the matching stage of next period, the cost to a firm from creating an additional vacancy is $k$. The benefit to a firm from creating an additional vacancy is $\eta(\theta_i(u), u) J(g_i(u))$, i.e., the probability of filling the vacancy $\eta(\theta_i(u), u)$ times the lifetime profit from employing an additional worker $J(g_i(u))$. A firm creates infinitely many vacancies if $\eta(\theta_i(u), u) J(g_i(u)) > k$, it creates no vacancies if $\eta(\theta_i(u), u) J(g_i(u)) < k$ and it is indifferent about the number of vacancies it creates if $\eta(\theta_i(u), u) J(g_i(u)) = k$. The equilibrium vacancy-to-unemployment ratio, $\theta_i(u)$, is consistent with the firm’s optimal vacancy creation strategy if and only if

$$\eta(\theta_i(u), u) J(g_i(u)) \leq k,$$

and $\theta_i(u) \geq 0$ with complementary slackness. Notice that the above condition implies that the market tightness $\theta_i(u)$ is given by

$$\theta_i(u) = \eta^{-1}\left(\min\left\{\frac{k}{J(g_i(u))}, 1\right\}, u\right).$$

The measure of workers who are unemployed at the bargaining stage of next period is

$$g_i(u) = u(1 - \varphi(J(g_i(u)), u)) + (1 - u) [q(e(u)) d_iH(u) + (1 - q(e(u))) d_iL(u)],$$

where $\varphi(J(g_i(u)), u) = \lambda(\theta_i(u), u)$ and $\theta_i(u)$ is the function of $J(g_i(u))$ and $u$ in (2.9). The first term on the right-hand side of (2.10) is the measure of workers who are unemployed at the bargaining stage of the current period, $u$, times the worker’s probability of not finding a job during the matching stage of next period, $1 - \varphi(J(g_i(u)), u)$. The second term on the right-hand side of (2.10) is the measure of workers who are employed at the bargaining stage of the current period, $1 - u$, times the worker’s probability of being fired during the separation stage of next period, $q(e(u)) d_iH(u) + (1 - q(e(u))) d_iL(u)$.
We are now in the position to define a recursive equilibrium for our model economy.

**Definition 1:** A Recursive Equilibrium is a tuple \((V, J, x, g_i)\) such that the equations (2.4), (2.6), (2.7) and (2.10) are satisfied for all \(u \in [0, 1]\) and \(i = \{B, G\}\).

### 3 Optimal Contract

In this section, we characterize the outcome of the contractual bargaining problem (2.7) between the worker and the firm. Throughout the analysis, we will assume that in the current period there are gains from trade between the firm and the worker and, hence, a solution to the bargaining problem exists. Similarly, we will assume that there are gains from trade in both of next period’s states of the world. In order to reduce notation, we will use \(F(x)\) and \(G(x)\) as shorthand for \(F(x, u)\) and \(G(x, u)\), \(V_i\) and \(J_i\) as shorthand for \(V(g_i(u))\) and \(J(g_i(u))\), and \(q_1\) and \(q_0\) as shorthand for \(q(1)\) and \(q(0)\).

We start by characterizing the properties of the solution to the Nash problem (2.7) under the assumption that the solution is a contract that requires the worker to exert effort. The main result here is that the contract prescribes that the worker should be fired when his production is not successful and next period’s state of the world is such that the gains from trade accruing to the worker are highest relative to those accruing to the firm. Next, we identify sufficient conditions under which the solution to (2.7) is a contract that requires the worker to exert effort. The main result here is that the contract requires the worker to exert effort if the additional output from putting in effort is sufficiently large relative to the disutility of effort.

The first lemma states that, if a contract solves the Nash problem (2.7) and requires the worker to exert effort, then the contract is such that the worker’s incentive compatibility constraint binds with equality. This result is intuitive. Providing incentives to the worker is costly as it requires severing a relationship with positive gains from trade. In order to minimize this cost, the optimal contract gives the worker just enough incentives to exert effort.

**Lemma 3.1.** Let \(x = (w, e, d_{iL}, d_{iH})\) be a solution to the problem (2.7) with \(e = 1\). The contract \(x\) is such that the worker’s incentive compatibility constraint binds, i.e.,

\[
  c = \beta(p_1 - q_0) \sum_{i = \{B, G\}} \pi_i(d_{iL} - d_{iH})V_i.
\]

**Proof:** On the way to a contradiction, suppose that \(x\) is such that

\[
  c > \beta(q_1 - q_0) \sum_{i = \{B, G\}} \pi_i(d_{iL} - d_{iH})V_i.
\]
Hence, $d_{iL} > 0$ for some $i = \{B, G\}$. Suppose without loss in generality that $d_{BL} > 0$.

Now, consider an alternative contract $x' = (w', e', d'_{iL}, d'_{iH})$ such that $w' = w$, $e' = e$, $d'_{iH} = d_{iH}$ for $i = \{B, G\}$, $d'_{GL} = d_{GL}$, $d'_{BL} = d_{BL} - \epsilon$ for some $\epsilon \in (0, d_{BL})$ small enough to satisfy the worker’s incentive compatibility constraint. By construction, the contract $x'$ is feasible. Moreover, the difference between the gains from trade accruing to the firm under contract $x'$ and under contract $x$ is given by

$$F(x') - F(x) = \beta \pi_B (1 - q_1) J_B \epsilon > 0.$$  

Similarly, the difference between the gains from trade accruing to the worker under contract $x'$ and under contract $x$ is given by

$$G(x') - G(x) = \beta \pi_B (1 - q_1) V_B \epsilon > 0.$$  

From the above inequalities, it follows that

$$F(x')^\gamma G(x')^{1-\gamma} > F(x)^\gamma G(x)^{1-\gamma},$$

which contradicts the assumption that $x$ is a solution to (2.7).  

The second lemma shows that, if a contract solves the Nash problem (2.7) and it requires the worker to exerting effort, then the contract is such that the worker is never fired if production is successful. This result is also easy to understand. Firing the worker after successful production has no benefit and two costs. First, doing so weakens the worker’s incentive to put in effort, as production is more likely to succeed when effort is positive. Second, doing so lowers the value of the match, as it leads to the destruction of the match when the gains from trade are positive. Hence, the optimal contract never involves firing if realized output is high.

**Lemma 3.2.** Let $x = (w, e, d_{iL}, d_{iH})$ be a solution to the problem (2.7) with $e = 1$ The contract $x$ is such that the worker is never fired if production is successful, i.e.,

$$d_{iH} = 0, \text{ for } i = \{B, G\}. \tag{3.2}$$

**Proof:** On the way to a contradiction, suppose that $x$ is such that $d_{iH} = 0$ for some $i = \{B, G\}$. Suppose without loss in generality that $d_{BH} > 0$.

Now consider an alternative contract $x' = (w', e', d'_{iL}, d'_{iH})$ such that $w' = w$, $e' = e$, $d'_{iL} = d_{iL}$ for $i = \{B, G\}$, $d'_{GH} = d_{GH}$ and $d'_{BH} = 0$. Clearly, the contract $x'$ is feasible. Moreover, the difference between the gains from trade accruing to the firm under contract $x'$ and under contract $x$ is given by

$$F(x') - F(x) = \beta \pi_B q_1 d_{BH} J_B > 0.$$  

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Similarly, the difference between the gains from trade accruing to the worker under contract $x'$ and under contract $x$ is given by

$$G(x') - G(x) = \beta \pi B q_1 d_{BH} V_B > 0.$$  

From the above inequalities, it follows that

$$F(x')^\gamma G(x')^{1-\gamma} > F(x)^\gamma G(x)^{1-\gamma},$$

which contradicts the assumption that $x$ is a solution to (2.7). ■

The next lemma is one of the central results of the paper. Suppose that, through some general equilibrium effects, there is a realization of the sunspot, say $k \in \{B, G\}$, for which the gains from trade accruing to the worker relative to those accruing to the firm, i.e. $V_k/J_k$, are larger than for the other realization of the sunspot, say $j$. The lemma shows that, if a contract solves the Nash problem (2.7) and involves the worker exerting effort, then the contract provides incentives to the worker by placing the highest possible firing probability (conditional on the incentive compatibility constraint being binding) on the state of the world where production is unsuccessful and the realization of the sunspot is $k$, i.e. the realization where the worker’s cost from losing the job is high relative to the firm’s cost from losing the worker. If the threat of being fired with probability 1 in the state of the world where the output is $y_L$ and the sunspot is $k$ does not provide the worker with enough incentives to exert effort, the contract also places a positive firing probability on the state of the world where production is unsuccessful and the realization of the sunspot is $j$, i.e. the realization where the worker’s cost from losing the job is low relative to the firm’s cost from losing the worker.

**Lemma 3.3.** Let $x = (w, e, d_{iL}, d_{iH})$ be a solution to the problem (2.7) with $e = 1$. If $V_k/J_k$ is strictly greater than $V_j/J_j$ for $k, j \in \{B, G\}$ and $k \neq j$, the contract $x$ is such that the worker is fired after unsuccessful production with highest probability in state $k$. That is, $d_{kL}$ is given by

$$d_{kL} = \min \left\{ \frac{c}{\beta(q_1 - q_0) \pi_k V_k}, 1 \right\}, \quad (3.3)$$

and $d_{jL}$ is given by

$$d_{jL} = \max \left\{ 0, \left[ \frac{c}{\beta(q_1 - q_0) - \pi_k V_k} - \pi_j V_j \right] / \pi_j V_j \right\}. \quad (3.4)$$

**Proof:** Suppose the contract $x$ is such that $d_{kL} < 1$ and $d_{jL} > 0$. From Lemma 3.2, it follows that $x$ is such that $d_{iH}$ equals 0 for $i = \{B, G\}$. From Lemma 3.1, it follows that $\beta(q_1 - q_0) \sum \pi_i d_{iL} V_i$ equals $c$.

Now, consider an alternative contract $x' = (w', e', d'_{iH}, d'_{iL})$ such that $w' = w$, $e' = e$,
\[ d'_i H = 0 \text{ for } i = \{B, G\}, \quad d'_k L = d_k L + \epsilon, \quad \epsilon \in (0, 1 - d_k L), \quad \text{and} \]
\[ d'_j L = d_j L - \frac{\pi_k V_k}{\pi_j V_j} \epsilon. \]

It is straightforward to verify that the contract \( x' \) is feasible. The difference between the gains from trade accruing to the firm under contract \( x' \) and under contract \( x \) is given by

\[
F(x') - F(x) = \beta(1 - q_1) \left[ \pi_k (d_k L - d'_k L) J_k + \pi_j \left( d_j L - d'_j L \right) J_j \right] \\
= \beta(1 - q_1) \epsilon \left[ -\pi_k J_k + \pi_k \frac{V_k}{V_j} J_j \right] > 0,
\]

where the last inequality follows from the fact that \( V_k / J_k \) is strictly greater than \( V_j / J_j \).

The difference between the gains from trade accruing to the worker under contract \( x' \) and under contract \( x \) is given by

\[
G(x') - G(x) = \beta(1 - q_1) \left[ \pi_k (d_k L - d'_k L) V_k + \pi_j \left( d_j L - d'_j L \right) V_j \right] \\
= \beta(1 - q_1) \epsilon \left[ -\pi_k V_k + \pi_k V_k \right] = 0.
\]

From the above inequalities, it follows that

\[
F(x')^\gamma G(x')^{1-\gamma} > F(x)^\gamma G(x)^{1-\gamma},
\]

which contradicts the assumption that \( x \) is a solution to (2.7). In turn, this implies that a contract \( x \) that solves (2.7) cannot be such that the worker is fired in the state \( j \), unless he is fired with probability 1 in state \( k \), which is the result that we needed to prove. ■

The result in Lemma 3 is intuitive. In order to induce the worker to exert effort, the contract must sometimes prescribe firing and, hence, the destruction of the firm-worker match when production is not successful. However, it is only the worker’s share of the value of the destroyed match that, ex-ante, provides the worker with an incentive to exert effort. The firm’s share of the value of the destroyed match is just “collateral damage.” The optimal contract provides the worker with the incentive to exert effort while minimizing the loss to the firm by assigning as high a firing probability as possible when the realization of the sunspot is such that the worker’s cost from losing the job (i.e. the worker’s gains from trade) are highest compared to the firm’s cost from losing the worker (i.e. the firm’s gains from trade).

Lemma 3 characterizes the destruction probabilities \( d_k L \) and \( d_j L \) when, due to general equilibrium effects, different realizations of the sunspot are associated with different ratios of gains from trade accruing to the worker relative to those accruing to the firm. When different realizations of the sunspot are not associated with different relative gains from trade, e.g. because firm-worker matches do not coordinate their behavior, the firing
probabilities $d_{kL}$ and $d_{jL}$ are not uniquely determined. For instance, $d_{kL}$ and $d_{jL}$ may be
given as in Lemma 3. Alternatively, and perhaps more naturally, $d_{kL}$ and $d_{jL}$ may be equal.

The next lemma provides a condition for the wage paid by the firm to the worker in any
contract that solves the Nash problem (2.7) and that requires the worker to exert effort.

**Lemma 3.4.** Let $x = (w, e, d_{iL}, d_{iH})$ be a solution to the problem (2.7) with $e = 1$. The wage $w$ is such that

$$
\gamma V = (1 - \gamma) v'(w) J,
$$

where $J$ and $V$ denote the value given by

$$
J = q_1 y_H + (1 - q_1) y_L - w + \beta \sum_{i = \{B,G\}} \pi_i [1 - (1 - q_1) d_{iL}] J_i,
$$

$$
V = v(w) - v(b) - c + \beta \sum_{i = \{B,G\}} \pi_i [1 - \varphi(J_i, u) - (1 - q_1) d_{iL}] V_i.
$$

**Proof:** The result follows from the first order condition of (2.7) with respect with the wage $w$. The expressions for $J$ and $V$ follow from Lemmas 3.2 and 3.3.

It is useful to briefly discuss the wage equation (3.5). The left-hand side of (3.5) are
the gains from trade accruing to the worker multiplied by the firm’s bargaining power and
the firm’s marginal disutility of paying a marginally higher wage. The right-hand side of
(3.5) are the gains from trade accruing to the firm multiplied by the worker’s bargaining
power and the worker’s marginal utility of being paid a marginally higher wage. The Nash
bargaining solution equates the left and the right hand side of (3.5).

The wage equation (3.5) is particularly important because it tells us that the gains from
trade accruing to the firm relative to those accruing to the worker are given by

$$
\phi \equiv \frac{J}{V} = \frac{\gamma}{1 - \gamma} \frac{1}{v'(w)}.
$$

Since the worker’s marginal utility is strictly decreasing in the wage $w$, (3.6) tells us the gains from trade accruing to the firm relative to those accruing to the worker are strictly increasing in $w$. Combining this result with Lemma 3, we can conclude that, if the contract
that solves the Nash product (2.7) requires the worker to exert effort, then the contract
will prescribe a higher firing probability for an unsuccessful worker in the state of the
world where his wage is lowest. This suggests that, if higher unemployment rate lowers the
worker’s value of being unemployed and, hence, the worker’s wage, then the contract will
prescribe a higher firing probability when the unemployment rate is higher. As we shall see
in the next section, this mechanism may lead to self-fulfilling unemployment fluctuations.

So far, we have assumed that the solution to the Nash problem (2.7) is a contract that
requires the worker to exert effort. The final lemma of this section shows that this is indeed
the case when the difference \((q_1 - q_0)(y_H - y_L)\) between the output produced by a worker who exerts effort and the output produced by a worker who does not exert effort is sufficiently large relative to the worker’s disutility of effort, \(c\).

**Lemma 3.5.** The solution to the Nash problem \((2.7)\) is some \(x = (w, e, d_{iH}, d_{iL})\) with \(e = 1\) if:

(i) When \(c \leq \beta(q_1 - q_0)\pi_k V_k\),

\[
(q_1 - q_0)(y_H - y_L) \geq \frac{1 - q_0}{q_1 - q_0} \frac{1}{V'} + \frac{1 - q_1}{q_1 - q_0} \frac{J_k}{V_k}.
\]  

(ii) When \(c > \beta(q_1 - q_0)\pi_k V_k\),

\[
(q_1 - q_0)(y_H - y_L) \geq \frac{1 - q_0}{q_1 - q_0} \frac{c}{V'}
\]

\[
+ \beta(1 - q_1) \left\{ \pi_k J_k + \left[ \frac{c}{\beta(q_1 - q_0)} - \pi_k V_k \right] \frac{J_j}{V_j} \right\}.
\]

**Proof:** It is straightforward to verify that the solution to the Nash problem \((2.7)\) subject to the worker not exerting effort is \(x = (w, 0, d_{iH}, d_{iL})\), with \(d_{iH} = 0, d_{iL} = 0\) for \(i = \{B, G\}\). Under contract \(x\), the gains from trade accruing to the firm are given by

\[
F(x) = q_0 y_H + (1 - q_0) y_L - w + \beta \sum_{i \in \{B, G\}} \pi_i J_i.
\]  

Similarly, the gains from trade accruing to the worker are given by

\[
G(x) = v(w) - v(b) + \beta \sum_{i \in \{B, G\}} \pi_i [1 - \varphi(J_i, u)] V_i.
\]

(i) We want to find sufficient conditions under which there is a contract \(x'\) that induces the worker to exert effort and that delivers a higher Nash product than the contract \(x\). First, consider the case in which \(c \leq \beta(q_1 - q_0)\pi_k V_k\) where \(k \in \{B, G\}\) denotes the realization of the sunspot where \(V_k/J_k\) is highest and \(j\) denotes the other realization of the sunspot. In this case, let \(x' = (w', e', d'_{iH}, d'_{iL})\), where \(e' = 1, d'_{iH}\) for \(i = \{B, G\}\), \(d'_{iL}\) is as in Lemma 3.3, and \(w'\) is such that

\[
v(w') = v(w) - c \frac{1 - q_0}{q_1 - q_0}.
\]  

It is immediate to verify that the contract \(x'\) induces the worker to exert effort. Therefore,
under the contract $x'$, the gains from trade accruing to the firm are given by

$$F(x') = q_1y_H + (1 - q_1)y_L - w' + \beta \sum_{i \in \{B,G\}} \pi_i [1 - (1 - q_1)d_{iL}'] J_i$$

$$= q_0y_H + (1 - q_0)y_L - w + \beta \sum_{i \in \{B,G\}} \pi_i J_i \quad (3.12)$$

$$+ (q_1 - q_0)(y_H - y_L) - (w' - w) - \frac{1 - q_1}{q_1 - q_0} \frac{J_k}{V_k},$$

where the second line makes use of the values of $d_{iL}'$ in Lemma 3.3. Similarly, under the contract $x'$, the gains from trade accruing to the worker are given by

$$G(x') = v(w') - v(b) - c + \beta \sum_{i \in \{B,G\}} \pi_i [1 - \varphi(J_i, u) - (1 - q_1)d_{iL}'] V_i$$

$$= v(w') - v(b) + \beta \sum_{i \in \{B,G\}} \pi_i [1 - \varphi(J_i, u)] V_i - \frac{1 - q_0}{q_1 - q_0} \frac{1}{c} \quad (3.13)$$

$$= v(w) - v(b) + \beta \sum_{i \in \{B,G\}} \pi_i [1 - \varphi(J_i, u)] V_i,$$

where the second line makes use of the values of $d_{iL}'$ in Lemma 3.3 and the third line makes use of the value of $w'$ in (3.11).

By comparing (3.12) and (3.13) with (3.9) and (3.10) and noting that $w' - w$ is smaller than $(1 - q_0)c/[(q_1 - q_0)\pi_k V_k]$, it follows that $F(x')^{-\gamma}G(x)^{1-\gamma}$ is smaller than $F(x')^{-\gamma}G(x'^{1-\gamma})$ as long as

$$(q_1 - q_0)(y_H - y_L) \geq c \left[ \frac{1 - q_0}{q_1 - q_0} \frac{1}{V_k} + \frac{1 - q_1}{q_1 - q_0} \right].$$

(ii) Now, consider the case in which $c > \beta(q_1 - q_0)\pi_k V_k$. In this case, let $x' = (w', e', d_{iH}', d_{iL}')$, where $e' = 1$, $d_{iH}' = 0$ for $i = \{B,G\}$, $d_{iL}'$ is as in Lemma 3, and $w'$ is as in (3.11). It is immediate to verify that the contract $x'$ induces the worker to exert effort. Therefore, under the contract $x'$, the gains from trade accruing to the firm are given by

$$F(x') = q_1y_H + (1 - q_1)y_L - w' + \beta \sum_{i \in \{B,G\}} \pi_i [1 - (1 - q_1)d_{iL}'] J_i$$

$$= q_0y_H + (1 - q_0)y_L - w + \beta \sum_{i \in \{B,G\}} \pi_i J_i + (q_1 - q_0)(y_H - y_L) - (w' - w)$$

$$- \beta(1 - q_1) \left\{ J_k + \frac{c}{\beta(q_1 - q_0) - \pi_k V_k} \right\} J_j, \quad (3.14)$$

where the second line makes use of the values of $d_{iL}'$ in Lemma 3.3. Similarly, under the
contract \( x' \), the gains from trade accruing to the worker are given by
\[
G(x') = v(w') - v(b) - c + \beta \sum_{i \in \{B, G\}} \pi_i [1 - \varphi(J_i, u) - (1 - q_1)d_{iL}] V_i \\
= v(w') - v(b) + \beta \sum_{i \in \{B, G\}} \pi_i [1 - \varphi(J_i, u)] V_i - \frac{1 - q_0}{q_1 - q_0} c \\
= v(w) - v(b) + \beta \sum_{i \in \{B, G\}} \pi_i [1 - \varphi(J_i, u)] V_i, 
\]
(3.15)

By comparing (3.14) and (3.15) with (3.9) and (3.10) and noting that \( w' - w \) is smaller than \( (1 - q_0)c / [(q_1 - q_0)u'] \), it follows that \( F(x)'G(x)'^{1-\gamma} \) is smaller than \( F(x)'G(x)'^{1-\gamma} \) as long as
\[
(q_1 - q_0)(y_H - y_L) \geq \frac{1 - q_0}{q_1 - q_0} \frac{c}{u'} \\
+ \beta(1 - q_1) \left\{ \pi_k J_k + \left[ \frac{c}{\beta(q_1 - q_0)} - \pi_k V_k \right] \frac{J_j}{V_j} \right\}. 
\]

This completes the proof of the lemma. ■

We are now in the position to summarize the properties of the optimal contract.

**Theorem 1**: If either condition (3.7) or (3.8) holds, then the contract \( x(u) \) given by \((w(u), e(u), d_{iL}(u), d_{iH}(u))\) that solves the Nash problem (2.7) is such that: (i) the worker exerts effort, \( e(u) = 1 \), (ii) the worker is paid the wage \( w(u) \) given by (3.6), (iii) the worker is never fired if production is successful, \( d_{iH}(u) = 0 \) for \( i = \{B, G\} \), (iv) the worker is fired with probability \( d_{iL}(u) \) given by (3.3) and (3.4) if production is unsuccessful.

Since the worker is not fired if production is successful, we can drop the subscript \( L \) from \( d_{iL} \) and refer to \( d_i \) as the probability with which the worker is fired if production is unsuccessful in the current period and the realization of the sunspot is \( i = \{B, G\} \) in the next period. Also, notice that the probability that the worker is fired after unsuccessful production depends on \( u \) only through the effect of \( u \) on \( g_i(u) \) and, in turn, on \( V(g_i(u)) \) and \( J(g_i(u)) \). Hence, we can refer to \( d_i(g_i(u)) \) as the probability that the worker is fired if production is unsuccessful in the current period, the realization of the sunspot is \( i \) and the unemployment rate is \( g_i(u) \) in the next period.

### 4 Existence of Coordinated Equilibrium

In this section, we establish the existence of an equilibrium in which firms coordinate the firing of their workers based on the realization of the sunspot.
4.1 Structure of the existence proof

We define the function $\psi(u,k)$, where $u \in [0,1]$ and $k = \{1, 2, 3\}$. For $k = 1$, we denote $\psi(u,k)$ as $J(u)$, i.e. the gains from trade accruing to a firm from employing a worker. For $k = 2$, we denote $\psi(u,k)$ as $V(u)$, i.e. the gains from trade accruing to a worker from being employed by a firm. For $k = 3$, we denote $\psi(u,k)$ as $\phi(u)$, i.e. the effective bargaining power of the firm. We denote as $\Psi$ the set of functions $\psi : [0,1] \times \{1, 2, 3\} \rightarrow \mathbb{R}$ such that:

(1) For all all $u_0, u_1 \in [0,1]$, with $u_0 \leq u_1$, the difference $\psi(u_0, 1) - \psi(u_0, 1)$ is bounded between $D_J(u_1 - u_0)$ and $D_J(u_1 - u_0)$, the difference $\psi(u_0, 2) - \psi(u_0, 2)$ is bounded between $D_V(u_1 - u_0)$ and $D_V(u_1 - u_0)$ and the difference $\psi(u_0, 3) - \psi(u_0, 3)$ is bounded between $-D_\phi(u_1 - u_0)$ and $-D_\phi(u_1 - u_0)$, where $D_J \geq D_J \geq 0, D_V \geq D_V \geq 0, D_\phi \geq D_\phi \geq 0$ are constants; (2) For all $u \in [0,1]$, $\psi(u, 1)$ is bounded in $[J, \bar{J}], \psi(u, 2)$ is bounded in $[V, \bar{V}]$ and $\psi(u, 3)$ is bounded in $[\phi, \bar{\phi}]$, where $\bar{J} \geq J > 0, \bar{V} \geq V > 0, \bar{\phi} \geq \phi > 0$. In words, a function $\psi = (J, V, \phi)$ is in the set $\Psi$ if $J$ and $V$ are bounded and their "derivative" is positive and bounded, and if $\phi$ is bounded and its "derivative" is negative and bounded below. Following Menzio and Shi (2010), it is easy to verify that the set $\Psi$ is a non-empty, bounded, closed and convex subset of the space of bounded continuous functions on $[0,1] \times \{1, 2, 3\}$ with the sup norm.

We seek an equilibrium in which firms coordinate on firing workers when the realization of the sunspot is $B$. We take an arbitrary function $\psi = (J, V, \phi)$ from the set $\Psi$. Given $\psi$, we compute the law of motion for unemployment, $g_t$, using the equilibrium condition (2.10). Given $g_t$ and $\phi$, we show that unemployment is higher when the realization of the sunspot is $B$ and, in this state, the relative gains from trade accruing to the firm are low relative to those accruing to the workers. Hence, it is optimal for firms to concentrate their firing when the realization of the sunspot is $B$. Given the assumption, $\psi$ and $g_t$, we compute the bargained wage $w$, using the equilibrium condition (3.5). Finally, given the assumption, $\psi$, $g_t$ and $w$, we use the equilibrium conditions (2.6), (2.4) and (3.6) to compute updates for the gains from trade accruing to the firm, $J'$, for the gains from trade accruing to the worker, $V'$, and for the effective bargaining power of the firm, $\phi'$.

The procedure described above implicitly defines a mapping $T$ from $\psi = (J, V, \phi)$ into $\psi' = (J', V', \phi')$. We show that $J'$, $V'$ and $\phi'$ are bounded and continuous functions and their "derivatives" are bounded in, respectively, $[D_J, \bar{D}_J], [D_V, \bar{D}_V]$ and $[-D_\phi, -\bar{D}_\phi]$. Hence, $T$ maps functions $\psi \in \Psi$ into functions $\psi' \in \Psi$. We also show that $J'$, $V'$ and $\phi'$ are continuous with respect to the function $\psi$ with which they are computed. Hence, $T$ is a continuous map. Finally, since $T$, it follows that the family of functions $T(\Psi)$ is equicontinuous. Overall, $T$ satisfies the assumption of Schauder’s fixed point theorem (Theorem 17.4 in Stokey, Lucas and Prescott, 1989). Hence, there exists a $\psi^* = (J^*, V^*, \phi^*)$ such that $\psi^* = T\psi^*$. Given $\psi^* = (J^*, V^*, \phi^*)$, we apply the operator $T$ one more time and construct the associated
law of motion for unemployment, \( g^*_t \), the firing probability, \( d^*_t \) and the bargained wage, \( w^* \). Clearly, the tuple \((J^*, V^*, \phi^*, g^*_t, w^*, d^*_t)\) represents an equilibrium where firms find it optimal to coordinate on firing workers when the realization of the sunspot is \( B \).

In order to carry out the proof, it is useful to introduce some additional notation and some additional restrictions on the fundamentals. Let us denote as \( \varphi_u \) and \( \varphi_v \), the maximum and the minimum values of the negative of the partial derivative of \( \varphi(J, u) \) with respect to \( u \) across all \((u, J) \in [0, 1] \times [J, J] \). Similarly, let us denote as \( \varphi_J \) and \( \varphi_J \), the maximum and the minimum values of the partial derivative of \( \varphi(J, u) \) with respect to \( J \) across all \((u, J) \in [0, 1] \times [J, J] \). Given this notation, we set the Lipschitz bounds on \( V, J \) and \( \phi \) to

\[
D_V = 0, \quad D_V = \frac{\beta(1 + \varphi_u)(\tau' + \tau''J)}{\varphi'_u - \beta(1 + \varphi_u)(\tau' + \tau''J)} V\varphi_u, \tag{4.1}
\]

\[
D_J = 0, \quad D_J = \frac{\gamma}{1 - \gamma} \beta \left[D_V(1 + \varphi_u) + V\varphi_u\right] \tag{4.2}
\]

\[
D_\phi = 0, \quad D_\phi = \frac{\gamma^2 \tau''}{1 - \gamma^2} \beta \left[D_V(1 + \varphi_u) + V\varphi_u\right]. \tag{4.3}
\]

Since the Lipschitz bound \( D_V \) must be greater than \( D_V \), we need to impose the technical condition

\[
\varphi'_u - \beta(1 + \varphi_u)(\tau' + \tau''J) > 0. \tag{4.4}
\]

Clearly, the condition (4.4) is satisfied when \( \beta \) is sufficiently low.

Second, we impose the condition

\[
\varphi_u - \varphi_J(1 + \varphi_u) > \frac{(1 - \gamma)\tau'}{\gamma V} \left[D_J + \frac{1 - q_1}{q_1 - q_0} \frac{c}{\beta} \varphi\right] (1 + \varphi_u). \tag{4.5}
\]

The condition (4.5) is satisfied for \( c/\beta \) sufficiently low. The condition it guarantees that the congestion externality in the matching process caused by higher unemployment is strong enough to generate a negative relationship between the equilibrium job-finding rate and unemployment rate, as well as between the equilibrium wage and the unemployment rate.

Finally, we impose the conditions

\[
\frac{c}{\beta} \leq (q_1 - q_0)\pi_B V, \tag{4.6}
\]

\[
(q_1 - q_0)(y_H - y_L) \geq c \left[ \frac{1 - q_0}{q_1 - q_0} \frac{1}{\varphi} + \frac{1 - q_1}{q_1 - q_0} \frac{1}{\varphi} \right].
\]

The conditions (4.6) are satisfied for \( c/\beta \) sufficiently low and \( y_H - y_L \) sufficiently high. As observed in Lemma 3.5, these conditions guarantee that the contract between a firm and a worker prescribes the worker to exert effort.
4.2 Law of motion for unemployment

We look for an equilibrium in which firms set the firing probability in the state of the world where the realization of the sunspot is \( B \) so as to satisfy the ex-ante incentive compatibility constraint of the worker, and set a firing probability of zero in the state of the world where the realization of the sunspot is \( G \). Now, take an arbitrary \( \psi = (J, V, \phi) \in \Psi \). Given \( \psi \), the law of motion for unemployment, \( g_i(u) \), satisfies the equilibrium condition (2.10) if and only if, for all \( u \in [0, 1] \), \( g_i(u) = u' \) where \( u' \) is such that

\[
u' = (1 - u)(1 - q_1)d_i(u') + u\left[1 - \varphi(J(u'), u)\right], \tag{4.7}\]

and

\[
d_B(u') = \left[\beta(q_1 - q_0)\pi_B V(u')\right]^{-1} c, \quad d_G(u') = 0. \tag{4.8}\]

First, notice that the left-hand side of (4.7) takes the value 0 for \( u' = 0 \), the value 1 for \( u' = 1 \) and is strictly increasing in \( u' \) for all \( u' \in [0, 1] \). The right-hand side of (4.7) takes a positive value for \( u' = 0 \), takes a positive value for \( u' = 1 \) and is decreasing in \( u' \) for all \( u' \in [0, 1] \). From this properties, it follows that there exists one and only one \( u' \in [0, 1] \) that solves (4.7). Therefore, there exists a unique law of motion for unemployment \( g_i \) that satisfies (2.10). Second, notice that the left-hand side of (4.7) is independent of \( u \), while the derivative of the right-hand side of (4.7) with respect to \( u \) is greater than \( q_1 - \varphi \), which is strictly positive by assumption. Therefore, the law of motion for unemployment \( g_i \) is strictly increasing in \( u \). Third, notice that the left-hand side of (4.7) is independent of the realization of the sunspot \( i = \{B, G\} \), while the right-hand side of (4.7) is strictly higher for \( i = B \) than for \( i = G \) for all \( u \in [0, 1] \), as \( d_B(u') > d_G(u') = 0 \) for all \( u = [0, 1] \). Therefore, the law of motion for unemployment is such that \( g_B(u) \geq g_G(u) \) for all \( u \in [0, 1] \) and \( g_B(1) = g_G(1) \).

The last property of \( g_i \) implies that \( \phi(g_B(u)) \leq \phi(g_G(u)) \), with strict inequality whenever \( u \in [0, 1] \) and \( \phi(u') \) is strictly decreasing in \( u' \). In light of Lemma 3.3 and of condition (4.6), \( \phi(g_B(u)) \leq \phi(g_G(u)) \) implies that there is a contract that maximizes the Nash product between an individual firm and a worker—taking as given the law of motion for unemployment \( g_i \) and the future effective relative bargaining power of the firm \( \phi \)—with the property the firing probabilities are \( d_B(u') \) and \( d_G(u') \) specified in (4.8). Moreover, when \( \phi(g_B(u)) \) \( < \phi(g_G(u)) \), this is the unique optimal contract.

Next, we want to prove that the law of motion for unemployment \( g_i \) is continuous in \( u \) and its “derivative” is positive and bounded above.

**Lemma 4.1:** For \( i = \{B, G\} \) and for all \( u_0, u_1 \in [0, 1] \) with \( u_0 < u_1 \), the law of motion for
unemployment, \(g_i(u)\), is such that

\[
\begin{align*}
D_g (u_1 - u_0) &< g_i(u_1) - g_i(u_0) \leq D_g (u_1 - u_0), \\
D_g &= 0, \quad D_g = 1 + \varphi_u.
\end{align*}
\] (4.9)

**Proof:** Let \(u'_1\) denote the solution to (4.7) for \(u = u_1\) and let \(u'_0\) denote the solution to (4.7) for \(u = u_0\), i.e.

\[
\begin{align*}
u'_1 &= (1 - u_1)(1 - q_1)d_i(u'_1) + u_1 [1 - \varphi(J(u'_1), u_1)], \\
u'_0 &= (1 - u_0)(1 - q_1)d_i(u'_0) + u_0[1 - \varphi(J(u'_0), u_0)].
\end{align*}
\]

Subtracting the second equation from the first one, we obtain

\[
u'_1 - u'_0 = u_1 - u_0 + (1 - q_1)(1 - u_1) [d_i(u'_1) - d_i(u'_0)] + (1 - q_1) [(1 - u_1) - (1 - u_0)] d_i(u'_0) + u_0 [\varphi(J(u'_0), u_0) - \varphi(J(u'_1), u_0)] + u_0 [\varphi(J(u'_1), u_0) - \varphi(J(u'_1), u_1)] + u_0 \varphi(J(u'_1), u_1) - u_1 \varphi(J(u'_1), u_1).\] (4.10)

Since \(V(u)\) is decreasing in \(u\) and \(u'_0 < u'_1\), \(d_i(u'_1) \leq d_i(u'_0)\). Hence, the second term on the right-hand side of (4.10) is non-positive. Since \(u_0 < u_1\), the third term on the right-hand side of (4.10) is non-positive. Since \(\varphi(J,u)\) is increasing in \(J\), \(J(u)\) is increasing in \(u\) and \(u'_0 < u'_1\), the fourth term on the right-hand side of (4.10) is non-positive. Finally, since \(u_0 < u_1\), the last term on the right-hand side of (4.10) is also non-positive. Hence, an upper bound on \(u'_1 - u'_0\) is given by

\[
u'_1 - u'_0 \leq u_1 - u_0 + u_0 [\varphi(J(u'_1), u_0) - \varphi(J(u'_1), u_1)] \leq [1 + \varphi_u](u_1 - u_0).
\]

Combining the above inequality with \(u'_0 < u'_1\), we obtain

\[
D_g (u_1 - u_0) < u'_1 - u'_0 \leq D_g (u_1 - u_0),
\]

where

\[
D_g = 0, \quad D_g = 1 + \varphi_u. \quad \blacksquare
\]

The law of motion for unemployment \(g_i(u)\) depends on the arbitrary function \(\psi\). Consider two arbitrary functions \(\psi_0 = (J_0, V_0, \phi_0)\), \(\psi_1 = (J_1, V_1, \phi_1)\) with \(\psi_0, \psi_1 \in \Theta\). Let \(g_{i0}\) denote the law of motion computed using \(\psi_0\) and let \(g_{i1}\) denote the law of motion computed using \(\psi_1\). In the following lemma, we prove that, if the distance between \(\psi_0\) and \(\psi_1\) goes to
zero, so does the distance between \( g_{i0} \) and \( g_{i1} \). That is, the law of motion for unemployment, \( g_i \), is continuous in the arbitrary function \( \psi \) with which it is computed.

**Lemma 4.2.** For any \( \delta > 0 \) and any \( \psi_0, \psi_1 \in \Theta \), if \( \|\psi_0 - \psi_1\| < \delta \), then

\[
\|g_{i0} - g_{i1}\| < \alpha g \delta, \quad \alpha g = \frac{c}{\beta(q_1 - q_0)\pi_B V^2 + \varphi'}, \tag{4.11}
\]

**Proof:** Take an arbitrary \( u \in [0,1] \). Let \( u'_0 \) denote \( g_{i0}(u) \) and \( u'_1 \) denote \( g_{i1}(u) \). From (4.7), it follows that

\[
u'_0 = (1 - u)(1 - q_1)d_{i0}(u'_0) + u[1 - \varphi(J_0(u'_0), u)], \]

\[
u'_1 = (1 - u)(1 - q_1)d_{i1}(u'_1) + u[1 - \varphi(J_1(u'_1), u)].
\]

Without loss in generality suppose that \( u'_0 \geq u'_1 \). In this case,

\[
\begin{align*}
u'_0 - u'_1 &= (1 - u)(1 - q_1)\{[d_{i0}(u'_0) - d_{i0}(u'_1)] + [d_{i0}(u'_1) - d_{i1}(u'_1)]}\]
&+ u\{[\varphi(J_1(u'_1), u) - \varphi(J_1(u'_0), u)] + [\varphi(J_1(u'_0), u) - \varphi(J_0(u'_0), u)]\}.
\end{align*}
\]

The term \( d_{i0}(u'_0) - d_{i0}(u'_1) \) is negative as \( d_{i0}(u') \) is decreasing in \( u' \) and \( u'_0 \geq u'_1 \). The term \( \varphi(J_1(u'_1), u) - \varphi(J_1(u'_0), u) \) is negative as \( \varphi(J, u) \) is increasing in \( J \), \( J_1(u') \) is increasing in \( u \) and \( u'_0 \geq u'_1 \). Therefore, we have

\[
u'_0 - u'_1 \leq (1 - u)(1 - q_1)[d_{i0}(u'_1) - d_{i1}(u'_1)] + u[\varphi(J_1(u'_0), u) - \varphi(J_0(u'_0), u)]
< \frac{c}{\beta(q_1 - q_0)\pi_B V^2 + \varphi'} \delta = \alpha g \delta.
\]

Since the above inequality holds for any \( u \in [0,1] \), we conclude that \( \|g_{i0} - g_{i1}\| < \alpha g \delta \). \[\blacksquare\]

### 4.3 Gains from trade

Given \( \psi = (J, V, \phi) \) and \( g_i \), we can compute the gains from trade accruing to a firm from employing a worker given that the contract specifies an arbitrary wage \( w \) and the optimal firing probability \( d_i(g_i(u)) \). We denote as \( F(w, u) \) these gains from trade which are given by

\[
F(w, u) = q_1 y_H + (1 - q_1) y_L - w - \frac{1 - q_1}{q_1 - q_0} \phi(g_i(u)) + \beta \sum_{i=\{B,C\}} \pi_i J(g_i(u)). \tag{4.12}
\]

Similarly, we can compute the gains from trade accruing to a worker from being employed by a firm given that the contract specifies an arbitrary wage \( w \) and the optimal firing probability.
Lemma 4.3: We denote as $G(w, u)$ these gains from trade, which are given by

$$G(w, u) = v(w) - v(b) - \frac{1 - q_0 c}{q_1 - q_0} + \beta \sum_{i \in \{B, G\}} \pi_i (1 - \varphi(J(g_i(u)), u)) V(g_i(u)). \tag{4.13}$$

In the next lemma, we show that $F$ is continuous and increasing in $u$ and its derivative with respect to $u$ is bounded above.

**Lemma 4.4:** For all $w \in \mathbb{R}$ and all $u_0, u_1 \in [0, 1]$ with $u_0 < u_1$, the value $\tilde{F}(w, u)$ to a firm from employing a worker at the wage $w$ is such that

$$\frac{D_F}{D_F} (u_1 - u_0) \leq F(w, u_1) - F(w, u_0) \leq \frac{D_F}{D_F} (u_1 - u_0),$$

$$\frac{D_F}{D_F} = 0, \quad \frac{D_F}{D_F} = \left[ \beta \frac{D_J}{D_J} + \frac{1 - q_1 c D_{\phi}}{q_1 - q_0} \right] \frac{D_g}{D_g}. \tag{4.14}$$

**Proof:** The difference between $F(w, u_1)$ and $F(w, u_0)$ is given by

$$F(w, u_1) - F(w, u_0) = \beta \sum_{i \in \{B, G\}} \pi_i [J(g_i(u_1)) - J(g_i(u_0))] + \frac{1 - q_1 c}{q_1 - q_0} \phi (g_B(u_0)) - \phi (g_B(u_1)).$$

Notice that $J(g_i(u_1)) - J(g_i(u_0))$ is positive as $J(u)$ is increasing in $u$, $g_i(u)$ is increasing in $u$ and $u_0 < u_1$. Moreover, $J(g_i(u_1)) - J(g_i(u_0))$ is smaller than $\frac{D_J}{D_J} (u_1 - u_0)$. Similarly, notice that $\phi (g_B(u_0)) - \phi (g_B(u_1))$ is positive as $\phi (u)$ is decreasing in $u$, $g_i(u)$ is increasing in $u$ and $u_0 < u_1$. Moreover, $\phi (g_B(u_0)) - \phi (g_B(u_1))$ is smaller than $\frac{D_J}{D_J} (u_1 - u_0)$.

From the above observations, it follows that

$$0 \leq F(w, u_1) - F(w, u_0) \leq \left[ \beta \frac{D_J}{D_J} + \frac{1 - q_1 c D_{\phi}}{q_1 - q_0} \right] \frac{D_g}{D_g} (u_1 - u_0).$$

The desired result follows from letting

$$\frac{D_F}{D_F} = 0 \text{ and } \frac{D_F}{D_F} = \left[ \beta \frac{D_J}{D_J} + \frac{1 - q_1 c D_{\phi}}{q_1 - q_0} \right] \frac{D_g}{D_g}. \quad \blacksquare$$

The next lemma shows that $G$ is continuous and strictly increasing in $u$ and its derivative with respect to $u$ is bounded away from zero and is bounded above.

**Lemma 4.4:** For all $w \in \mathbb{R}$ and all $u_0, u_1 \in [0, 1]$ with $u_0 < u_1$, the value $\tilde{G}(w, u)$ to a worker from being employed at the wage $w$ is such that

$$\frac{D_G}{D_G} (u_1 - u_0) < G(w, u_1) - G(w, u_0) \leq \frac{D_G}{D_G} (u_1 - u_0),$$

$$\frac{D_G}{D_G} = \frac{\varphi}{\varphi_J} \frac{D_{\varphi}}{D_{\varphi}} > 0, \quad \frac{D_G}{D_G} = \beta \left[ \frac{D_{\varphi}}{D_{\varphi}} + \frac{D_{\varphi}}{D_{\varphi}} \right]. \tag{4.15}$$
Proof: The difference between $G(w, u_1)$ and $G(w, u_0)$ is given by

$$G(w, u_1) - G(w, u_0) = \beta \sum_{i \in \{B, G\}} \pi_i \left[ V(g_i(u_1)) - V(g_i(u_0)) \right] (1 - \varphi(J(g_i(u_0)), u_0))$$

$$+ \beta \sum_{i \in \{B, G\}} \pi_i V(g_i(u_1)) \left[ \varphi(J(g_i(u_0)), u_0) - \varphi(J(g_i(u_0)), u_1) \right]$$

$$+ \beta \sum_{i \in \{B, G\}} \pi_i V(g_i(u_1)) \left[ \varphi(J(g_i(u_0)), u_1) - \varphi(J(g_i(u_1)), u_1) \right].$$

The first term on the right-hand side of (4.16) is positive as $V(u)$ is increasing in $u$, $g_i(u)$ is increasing in $u$ and $u_0 < u_1$. Moreover, this term is smaller than $\beta \varphi_J \varphi_u (u_1 - u_0)$.

The sum of the second and third term on the right-hand side of (4.16) is positive as $\varphi_J \varphi_u (u_1 - u_0)$ and smaller than $\beta \varphi_J \varphi_u (u_1 - u_0)$.

From the above observations, it follows that

$$\beta \varphi_J \varphi_u (u_1 - u_0)$$

$$\leq G(w, u_1) - G(w, u_0)$$

$$\leq \beta \varphi_J \varphi_u (u_1 - u_0).$$

The desired result follows from letting

$$\varphi_J = \beta \varphi_J \varphi_u (u_1 - u_0)$$

and

$$\varphi_J = \beta \varphi_J \varphi_u (u_1 - u_0).$$

Now, consider the two arbitrary functions $\psi_0$ and $\psi_1$. Let $F_0$ and $G_0$ denote the gains from trade accruing to the firm and to the worker computed using $g_i(0)$ and $\psi_0$. Similarly, let $F_1$ and $G_1$ denote the gains from trade accruing to the firm and to the worker computed using $g_i(1)$ and $\psi_1$. In the following lemma, we prove that, if the distance between $\psi_0$ and $\psi_1$ goes to zero, so does the distance between $F_0$ and $F_1$ and between $G_0$ and $G_1$. That is, the value of employment to the firm and to the worker are continuous in the arbitrary function $\psi$.

Lemma 4.5: For any $\delta > 0$ and any $\psi_0, \psi_1 \in \Theta$, if $\|\psi_0 - \psi_1\| < \delta$, then

$$\|F_0 - F_1\| < \alpha_F \delta,$$

$$\alpha_F = \left[ 1 + \varphi_J \varphi_u \right] + \frac{1 - q_1}{q_1 - q_0} c \left[ 1 + \varphi_J \varphi_u \right],$$

and

$$\|G_0 - G_1\| < \alpha_G \delta,$$

$$\alpha_G = \left[ 1 + \varphi_J \varphi_u \right] + \varphi_J \varphi_u \left[ 1 + \varphi_J \varphi_u \right].$$

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\textbf{Proof}: Take an arbitrary }u \in [0,1]. \text{ From (4.12), it follows that}
\begin{align*}
|F_0(w, u) - F_1(w, u)| \leq \beta \sum_{i = \{B, G\}} \pi_i \{ |J_0(g_{i0}(u)) - J_0(g_{i1}(u))| + |J_0(g_{i1}(u)) - J_1(g_{i1}(u))| \} + \frac{1 - q_1}{q_1 - q_0} \left[ |\phi_1(g_{i1}(u)) - \phi_1(g_{i0}(u))| + |\phi_1(g_{i0}(u)) - \phi_0(g_{i0}(u))| \right].
\end{align*}
\tag{4.19}

The term \(|J_0(g_{i0}(u)) - J_0(g_{i1}(u))|\) is smaller than \(D_j \alpha_g \delta\). The term \(|J_0(g_{i1}(u)) - J_1(g_{i1}(u))|\) is strictly smaller than \(\delta\). The term \(|\phi_1(g_{i1}(u)) - \phi_1(g_{i0}(u))|\) is smaller than \(D_g \alpha_g \delta\). Finally, the term \(|\phi_1(g_{i0}(u)) - \phi_0(g_{i0}(u))|\) is strictly smaller than \(\delta\). Therefore, we have
\begin{align*}
|F_0(w, u) - F_1(w, u)| < [1 + D_j \alpha_g] \delta + \frac{1 - q_1}{q_1 - q_0} [1 + D_g \alpha_g] \delta = \alpha_F \delta.
\end{align*}

From (4.13), it follows that
\begin{align*}
|G_0(w, u) - G_1(w, u)| \leq \beta \sum_{i = \{B, G\}} \pi_i (1 - \varphi(J(g_{i0}(u)), u)) \left\{ |V_0(g_{i0}(u)) - V_0(g_{i1}(u))| \right\} + |V_0(g_{i1}(u)) - V_1(g_{i1}(u))| \right\} + \beta \sum_{i = \{B, G\}} \pi_i V_1(g_{i1}(u)) \left\{ |\varphi(J_1(g_{i1}(u), u) - \varphi(J_0(g_{i1}(u)), u))| \right\}.
\end{align*}

The term \(|V_0(g_{i0}(u)) - V_0(g_{i1}(u))|\) is smaller than \(D_V \delta\). The term \(|V_0(g_{i1}(u)) - V_1(g_{i1}(u))|\) is strictly smaller than \(\delta\). The term \(|\varphi(J_1(g_{i1}(u), u) - \varphi(J_0(g_{i1}(u)), u))|\) is smaller than \(D_j \delta\). Finally, the term \(|\varphi(J_0(g_{i0}(u)), u) - \varphi(J_0(g_{i0}(u)), u)|\) is smaller than \(D_j D_g \alpha_g \delta\). Overall, we have
\begin{align*}
|G_0(w, u) - G_1(w, u)| < [1 + D_V] \delta + V \left[ D_j \delta + D_j D_g \alpha_g \delta \right] = \alpha_G \delta. \text{ Overall, we have}
\end{align*}
\tag{4.20}

Since (4.19) and (4.20) hold for any \(u \in [0,1]\), we conclude that \(\|F_0 - F_1\| < \alpha_F \delta\) and \(\|G_0 - G_1\| < \alpha_G \delta\). 

\subsection{4.4 Wage bargain}

Given \(\psi = (J, V, \phi), g_i, F\) and \(G\), the contract that maximizes the Nash product specifies a wage \(w(u)\) such that
\begin{align*}
(1 - \gamma)u'(w(u))F(w(u), u) = \gamma G(w(u), u).
\end{align*}
\tag{4.21}

Notice that the left-hand side of (4.21) is strictly decreasing in \(w\) as the worker’s marginal utility \(u'(w)\) is strictly decreasing in \(w\) and the firm’s gains from trade \(F(w, u)\) are strictly decreasing in \(w\). The right-hand side of (4.21) is strictly increasing in \(w\) as the worker’s value of employment \(G(w, u)\) is strictly increasing in \(w\). Therefore, for any \(u \in [0,1]\), there exists one and only one value of \(w\) that solves (4.21).
The next lemma shows that \( w \) is continuous and strictly decreasing in \( u \) and its “derivative” with respect to \( u \) is bounded below and above away from zero.

**Lemma 4.6:** For all \( u_0, u_1 \in [0, 1] \) with \( u_0 < u_1 \), the equilibrium wage \( w(u) \) is such that

\[
\begin{align*}
\frac{D_w}{w_0} (u_1 - u_0) &< w(u_0) - w(u_1) \leq \frac{D_w}{w_1} (u_1 - u_0), \\
\frac{D_w}{w_0} &= \left[ \frac{1}{1 - \gamma} + \frac{\psi''}{\psi'} \right]^{-1} \left[ \frac{\gamma D_G}{1 - \gamma} - \mathcal{D}_F \right] > 0, \\
\frac{D_w}{w_1} &= \frac{\gamma D_G}{1 - \gamma}.
\end{align*}
\]

(4.22)

**Proof:** To alleviate notation, let \( w_0 \) denote \( w(u_0) \) and \( w_1 \) denote \( w(u_1) \). First, we establish that \( w_0 > w_1 \). To this aim, notice that

\[
(1 - \gamma) v'(w_0) F(w_0, u_1) \\
\leq (1 - \gamma) v'(w_0) F(w_0, u_0) + (1 - \gamma) \psi \mathcal{D}_F (u_1 - u_0).
\]

Also, notice that

\[
\begin{align*}
\gamma G(w_0, u_1) \\
\geq \gamma G(w_0, u_0) + \gamma D_G (u_1 - u_0) \\
= (1 - \gamma) v'(w_0) F(w_0, u_0) + \gamma D_G (u_1 - u_0).
\end{align*}
\]

Since by assumption \((4.5)\) \( \gamma D_G (u_1 - u_0) \) is strictly greater than \( (1 - \gamma) \psi \mathcal{D}_F (u_1 - u_0) \), the right-hand side of \((4.21)\) is strictly greater than the left-hand side of \((4.21)\) when evaluated at \( w = w_0 \) and \( u = u_1 \). Since the right-hand of \((4.21)\) is strictly increasing in \( w \) and the left-hand side of \((4.21)\) is strictly decreasing in \( w \), it follows that \( w_1 \) is strictly smaller than \( w_0 \).

Next, we derive lower and upper bounds on the difference \( w_0 - w_1 \). To this aim, notice that

\[
(1 - \gamma) [v'(w_0) F(w_0, u_0) - v'(w_1) F(w_1, u_1)] = \gamma [G(w_0, u_0) - G(w_1, u_1)].
\]

The above equation can be rewritten as

\[
\begin{align*}
\frac{w_0 - w_1}{1 - \gamma} &= \frac{\gamma G(w_0, u_1) - G(w_0, u_0)}{v'(w_1)} - [F(w_1, u_1) - F(w_1, u_0)] \\
&\leq \frac{\gamma v(w_0) - v(w_1)}{v'(w_1)} - \frac{v'(w_1) - v'(w_0)}{v'(w_1)} F(w_0, u_0).
\end{align*}
\]

(4.23)

The term \( G(w_0, u_1) - G(w_0, u_0) \) is strictly positive, greater than \( D_G (u_1 - u_0) \) and smaller than \( D_G (u_1 - u_0) \). The term \( F(w_1, u_1) - F(w_1, u_0) \) is positive and smaller than \( D_F (u_1 - u_0) \). The term \( v(w_0) - v(w_1) \) is strictly positive, greater than \( v'(w_0) (w_0 - w_1) \) and smaller than \( v'(w_1) (w_0 - w_1) \). The term \( v'(w_1) - v'(w_0) \) is strictly positive, greater than \( v''(w_0 - w_1) \)
and smaller than $\pi''(w_0 - w_1)$.

From the above observations, it follows that

$$w_0 - w_1 \leq \left[ \frac{\gamma \, \frac{D_G}{1 - \gamma \, \frac{u'}{u'}}}{1 - \gamma \, \frac{u'}{u'}} \right] (u_1 - u_0).$$

Similarly, we have

$$w_0 - w_1 \geq \left[ \frac{1}{1 - \gamma + \frac{\pi''}{u'}} \right]^{-1} \left[ \frac{\gamma \, \frac{D_G}{1 - \gamma \, \frac{u'}{u'}} - D_F}{1 - \gamma \, \frac{u'}{u'}} \right] (u_1 - u_0).$$

The above inequalities represent the desired bounds on $w_0 - w_1$. ■

Now, consider the two arbitrary functions $\psi_0$ and $\psi_1$. Let $w_0$ denote the bargained wage computed using $F_0$, $G_0$, $g_0$ and $\psi_0$. Similarly, let $w_1$ denote the bargained wage computed using $F_1$, $G_1$, $g_1$ and $\psi_1$. In the following lemma, we prove that, if the distance between $\psi_0$ and $\psi_1$ goes to zero, so does the distance between $w_0$ and $w_1$. That is, the bargained wage is continuous in the arbitrary function $\psi$ with which it is computed.

Lemma 4.7: For any $\delta > 0$ and any $\psi_0, \psi_1 \in \Theta$, if $\|\psi_0 - \psi_1\| < \delta$, then

$$\|w_0 - w_1\| < \alpha_w \delta, \quad \alpha_w = \left[ 1 + \frac{1}{1 - \gamma + \frac{\pi''}{u'}} \right] + \frac{1 - q_1}{q_1 - q_0} c \left[ 1 + \frac{1}{1 - \gamma + \frac{\pi''}{u'}} \right]. \quad (4.24)$$

Proof: Take an arbitrary $u \in [0,1]$. To alleviate notation, let $w_0$ denote $w_0(u)$ and $w_1$ denote $w_1(u)$. From (4.21), it follows that

$$\gamma G_0(w_0, u) - (1 - \gamma) v'(w_0) F_0(w_0, u) = 0,$$
$$\gamma G_1(w_1, u) - (1 - \gamma) v'(w_1) F_1(w_1, u) = 0.$$ 

Subtracting the second equation from the first, we obtain

$$\gamma \{ G_0(w_0, u) - G_1(w_0, u) + v(w_0) - v(w_1) \}$$
$$+ (1 - \gamma)v'(w_0) \{ F_1(w_1, u) - F_0(w_1, u) + w_0 - w_1 \}$$
$$+ (1 - \gamma)F_1(w_1, u) [v'(w_1) - v'(w_0)] = 0.$$

Suppose without loss in generality that $w_0 - w_1 \geq 0$. Then, we can rewrite the previous equation as

$$w_0 - w_1 = F_0(w_1, u) - F_1(w_1, u) + F_1(w_1, u) \left[ \frac{v'(w_0) - v'(w_1)}{v'(w_0)} \right]$$
$$+ \frac{\gamma}{1 - \gamma} \left[ \frac{G_1(w_0, u) - G_0(w_0, u)}{v'(w_0)} + \frac{v(w_1) - v(w_0)}{v'(w_0)} \right].$$

The term $F_0(w_1, u) - F_1(w_1, u)$ is strictly smaller than $\alpha_F \delta$. The term $G_0(w_0, u) - G_1(w_0, u)$ is strictly smaller than $\alpha_G \delta$. The terms $v'(w_0) - v'(w_1)$ and $v(w_1) - v(w_0)$ are both non-
positive. Hence, we have

\[ 0 \leq w_0 - w_1 < \left[ \alpha_F + \frac{\gamma}{1 - \gamma} \frac{\alpha_G}{\alpha_F} \right] \delta = \alpha_w \delta. \]

Since the above inequality holds for any \( u \in [0, 1] \), we conclude that \( \|w_0 - w_1\| < \alpha_w \delta. \)

\section{4.5 Updated value functions}

Given \( \psi = (J, V, \phi), g_i, F, G \) and \( w \), we can compute an update \( J' \) for the gains from trade accruing to a firm given the optimal contract, an update \( V' \) for the gains from trade accruing to a worker given the optimal contract, and an update \( \phi' \) for the effective relative bargaining power of the firm. In particular, the updated gains from trade accruing to a firm given the optimal contract are given by

\[ J'(u) = F(w(u), u). \]

In the following lemma, we prove that the updated value function \( J' \) is such that its “derivative” is positive, bounded below away from zero and bounded above.

\begin{lemma}
For all \( u_0, u_1 \in [0, 1] \) with \( u_0 < u_1 \), the equilibrium value of a worker to a firm is such that

\[ \frac{D_{J'}}{D_{w'}} (u_1 - u_0) < J'(u_1) - J'(u_0) \leq \frac{\gamma}{1 - \gamma} \frac{D_{G}}{D_{w'}} (u_1 - u_0), \]

\[ D_{J'} = D_w, \quad D_{J'} = \frac{\gamma}{1 - \gamma} \frac{D_{G}}{D_{w'}}. \]
\end{lemma}

\begin{proof}
To simplify notation, let \( w_0 \) denote \( w(u_0) \) and \( w_1 \) denote \( w(u_1) \). The difference between \( J'(u_1) \) and \( J'(u_0) \) is given by

\[ J'(u_1) - J'(u_0) = F(w_1, u_1) - F(w_1, u_0) + w_0 - w_1. \]

From the above expression, it follows that

\[ J'(u_1) - J'(u_0) \geq D_{w'} (u_1 - u_0). \]

Using (xx) and (yy), we can write the difference between \( J'(u_1) \) and \( J'(u_0) \) as

\[ J'(u_1) - J'(u_0) = \frac{\gamma}{1 - \gamma} \frac{G(w_0, u_1) - G(w_0, u_0)}{v'(w_1)} - \frac{\gamma}{1 - \gamma} \frac{v(w_0) - v(w_1)}{v'(w_1)} F(w_0, u_0). \]

From the above expression and from the properties of \( G(w_0, u_1) - G(w_0, u_0), v(w_0) - v(w_1) \)
and \( u'(w_1) - u'(w_0) \) described in the proof of Lemma 4.6, it follows that
\[
J'(u_1) - J'(u_0) \leq \left[ \frac{\gamma}{1 - \frac{\beta}{\nu'}} \right] (u_1 - u_0). \tag{4.29}
\]
The inequalities in (4.28) and (4.29) establish the desired result.  

Notice that the updated value function \( J' \) satisfies Lipschitz bounds of the original value function \( J \) if 
\[
\overline{D}_J = \frac{1}{1 - \frac{\beta}{\nu'}} \overline{D}_{V'}(1 + \overline{V}_u) + \overline{V}_u. \tag{4.31}
\]
The first condition is satisfied because \( \overline{D}_{V'} > 0 \). The second condition is satisfied because, given \( D_{V'} \) defined in (4.1) and \( D_J \) defined in (4.2), \( \overline{D}_J = \overline{D}_{V'} \).

The value to the worker from being employed at the equilibrium contract is given by
\[
V'(u) = G(w(u), u). \tag{4.30}
\]
In the following lemma, we prove that the updated value function \( V' \) is such that its “derivative” is positive, bounded below away from zero and bounded above.

**Lemma 4.9:** For all \( u_0, u_1 \in [0, 1] \) with \( u_0 < u_1 \), the equilibrium value of employment to a worker is such that
\[
\overline{D}_{V'} (u_1 - u_0) < V'(u_1) - V'(u_0) \leq \overline{D}_V, (u_1 - u_0),
\]
\[
\overline{D}_{V'} = \frac{1 - \frac{\gamma}{\nu'} D_w}{\gamma}, \quad \overline{D}_V = \frac{\nu'}{\nu''} \overline{D}_G + \frac{1 - \frac{\gamma}{\nu''} \overline{D}_w}{\gamma}. \tag{4.31}
\]

**Proof:** To simplify notation, let \( w_0 \) denote \( w(u_0) \) and \( w_1 \) denote \( w(u_1) \). The difference between \( V'(u_1) \) and \( V'(u_0) \) is given by
\[
V'(u_1) - V'(u_0) = G(w_1, u_1) - G(w_1, u_0) + v(w_1) - v(w_0). \tag{4.32}
\]
Using (4.32) and (4.23), we can write the difference between \( V'(u_1) \) and \( V'(u_0) \) as
\[
V'(u_1) - V'(u_0) = \frac{1 - \frac{\gamma}{\nu'} v'(w_1) [J'(u_1) - J'(u_0)]}{\gamma} \\
+ \frac{1 - \frac{\gamma}{\nu''} [v'(w_1) - v'(w_0)]}{\gamma} F(w_0, u_0).
\]
From the above expression, it follows that
\[
V'(u_1) - V'(u_0) \leq \left[ \frac{\nu'}{\nu''} \overline{D}_G + \frac{1 - \frac{\gamma}{\nu''} \overline{D}_w}{\gamma} \right] (u_1 - u_0). \tag{4.33}
\]
Similarly, it follows that
\[ V'(u_1) - V'(u_0) \geq \left[ \frac{1 - \gamma' v'}{\gamma} w \right] (u_1 - u_0). \]

The above inequalities establish the desired result. \( \square \)

Notice that the updated value function \( V' \) satisfies Lipschitz bounds of the original value function \( V \) if \( D_{V'} \geq D_V \equiv 0 \) and
\[ D_{V'} \geq D_V \equiv \left[ \frac{v'}{v} + \frac{v''}{v'} \right] \beta [D_V (1 + \varphi_u) + V \varphi_u]. \]

The first condition is satisfied because \( D_{V'} > 0 \). The second condition is satisfied because, given \( D_V \) defined in (4.1), \( D_{V'} = D_V \).

The updated effective relative bargaining power of the firm is given by
\[ \phi'(u) = \frac{\gamma}{(1 - \gamma) v' w(u)}. \quad (4.33) \]

In the following lemma, we prove that the updated value function \( \phi' \) is such that its “derivative” is negative, bounded above away from zero and bounded below.

**Lemma 4.10**: For all \( u_0, u_1 \in [0, 1] \) with \( u_0 < u_1 \), the effective relative bargaining power of the firm is such that
\[ D_{\phi'} (u_1 - u_0) < \phi'(u_0) - \phi'(u_1) \leq D_{\phi'} (u_1 - u_0), \]
\[ D_{\phi'} = \frac{\gamma v''}{(1 - \gamma) v'^2} D_w, \quad D_{\phi'} = \frac{\gamma v''}{(1 - \gamma) v'^2} D_w. \quad (4.34) \]

**Proof**: To simplify notation, let \( w_0 \) denote \( w(u_0) \) and \( w_1 \) denote \( w(u_1) \). The difference between \( \phi'(u_0) \) and \( \phi'(u_1) \) is given by
\[ \phi'(u_0) - \phi'(u_1) = \frac{\gamma [v'(w_1) - v'(w_0)]}{(1 - \gamma) v'(w_0) v'(w_1)}. \]

From the above expression, it follows that
\[ \phi'(u_0) - \phi'(u_1) \geq \frac{\gamma v''}{(1 - \gamma) v'^2} D_w (u_1 - u_0). \]

Similarly, it follows that
\[ \phi'(u_0) - \phi'(u_1) \leq \frac{\gamma v''}{(1 - \gamma) v'^2} D_w (u_1 - u_0). \]

The above inequalities establish the desired result. \( \square \)

Notice that the updated function \( \phi' \) satisfies Lipschitz bounds of the original function \( \phi \)
if $D_{\phi'} \geq D_{\phi} = 0$ and
\[
D_{\phi} \geq D_{\phi'} = \frac{\gamma^2 \beta''}{(1-\gamma)^2} \beta \left[ D_V(1 + \varphi_u) + \nabla \varphi_u \right].
\]
The first condition is satisfied because $D_{\phi'} > 0$. The second condition is satisfied because, given $D_V$ defined as in (4.1) and $D_{\phi}$ defined in (4.3), $D_{\phi'} = D_{\phi}$.

Now, consider the two arbitrary functions $\psi_0$ and $\psi_1$. Let $J_0'$, $V_0'$ and $\phi_0'$ denote the updated value functions computed using $w_0$, $F_0$, $G_0$, $g_i0$ and $\psi_0$. Similarly, let $J_1'$, $V_1'$ and $\phi_1'$ denote the updated value functions computed using $w_1$, $F_1$, $G_1$, $g_i1$ and $\psi_1$. In the following lemma, we prove that, if the distance between $\psi_0$ and $\psi_1$ goes to zero, so does the distance between $J_0'$ and $J_1'$, between $V_0'$ and $V_1'$ and between $\phi_0'$ and $\phi_1'$. That is, the updated value functions are continuous in the arbitrary function $\psi$ with which they are computed.

**Lemma 4.11:** For any $\delta > 0$ and any $\psi_0, \psi_1 \in \Theta$, if $\| \psi_0 - \psi_1 \| < \delta$, then
\[
\begin{align*}
\| J_0' - J_1' \| &< \alpha_{J'} \delta, & &\alpha_{J'} = \alpha_w + \alpha_F, \\
\| V_0' - V_1' \| &< \alpha_{V'} \delta, & &\alpha_{V'} = \varphi' \alpha_w + \alpha_G, \\
\| \phi_0' - \phi_1' \| &< \alpha_{\phi'} \delta, & &\alpha_{\phi'} = \frac{\gamma}{1-\gamma} \frac{\varphi'' \alpha_w}{\varphi'}. 
\end{align*}
\]

**Proof:** Take an arbitrary $u \in [0, 1]$. The difference $J_0'(u) - J_1'(u)$ is such that
\[
|J_0'(u) - J_1'(u)| \leq |F_0(w_0(u), u) - F_0(w_1(u), u)| + |F_0(w_1(u), u) - F_1(w_1(u), u)| \\
= |w_0(u) - w_1(u)| + |F_0(w_1(u), u) - F_1(w_1(u), u)| \\
< \alpha_w \delta + \alpha_F \delta = \alpha_{J'} \delta.
\]
The difference $V_0'(u) - V_1'(u)$ is such that
\[
|V_0'(u) - V_1'(u)| \leq |G_0(w_0(u), u) - G_0(w_1(u), u)| + |G_0(w_1(u), u) - G_1(w_1(u), u)| \\
= |v(w_0(u)) - v(w_1(u))| + |G_0(w_1(u), u) - G_1(w_1(u), u)| \\
< \varphi' \alpha_w \delta + \alpha_G \delta = \alpha_{V'} \delta.
\]
Finally, the difference $\phi_0'(u) - \phi_1'(u)$ is such that
\[
|\phi_0'(u) - \phi_1'(u)| \leq \frac{\gamma}{1-\gamma} \left| \frac{1}{v'(w_0(u))} - \frac{1}{v'(w_1(u))} \right| \\
< \frac{\gamma}{1-\gamma} \frac{\varphi'' \alpha_w}{\varphi'} \delta = \alpha_{\phi'} \delta.
\]
Since the above inequalities hold for all $u \in [0, 1]$, we conclude that $\| J_0' - J_1' \| < \alpha_{J'} \delta$, $\| V_0' - V_1' \| < \alpha_{V'} \delta$ and $\| \phi_0' - \phi_1' \| < \alpha_{\phi'} \delta$.  

4.6 Existence

In the previous subsection, we established two key results. First, given any arbitrary \( \psi = (J, V, \phi) \in \Psi \), the updated value functions \( \psi' = (J', V', \phi') \) are bounded and such that for all \( u_0, u_1 \in [0, 1] \), with \( u_0 < u_1 \), the difference \( J(u_1) - J(u_0) \) is bounded between \( D_J \) and \( D_J' \), with \( D_J > D_J' \), the difference \( V(u_1) - V(u_0) \) is bounded between \( D_V \) and \( D_V' \), with \( D_V' > D_V \), and the difference \( \phi(u_0) - \phi(u_1) \) is bounded between \( D_\phi \) and \( D_\phi' \), with \( D_\phi' > D_\phi \) and \( D_\phi' = D_\phi \). This means that the operator \( T \) that maps functions \( \psi \in \Psi \) into functions \( \psi' \in \Psi \). Second, given any arbitrary \( \psi_0, \psi_1 \in \Psi \), \( \|\psi_0 - \psi_1\| < \delta \) implies that \( \|J'_0 - J'_1\| < \alpha_J \delta \), \( \|V'_0 - V'_1\| < \alpha_V \delta \) and \( \|\phi'_0 - \phi'_1\| < \alpha_\phi \delta \) and, hence, \( \|\psi'_0 - \psi'_1\| < \max\{\alpha_J, \alpha_V, \alpha_\phi\} \delta \). This means that the operator \( T \) is continuous in \( \psi \). Finally, it is immediate to see that the family of functions \( T(\Psi) \) is equicontinuous.

To see that this is the case let \( \|\cdot\|_E \) denote the standard norm on the Euclidean space \([0, 1] \times \{1, 2, 3\} \). For any \( \epsilon > 0 \), let \( \delta_\epsilon = \min\{\max\{D_J, D_V, D_\phi\}\}^{-1} \epsilon, 1\} \). Then, for all \( (u_0, k_0), (u_1, k_1) \in [0, 1] \times \{1, 2, 3\} \) such that \( \|(u_0, k_0) - (u_1, k_1)\|_E < \delta_\epsilon \), we have

\[
\begin{align*}
\quad &|T(\psi)(u_0, k_0) - (\psi')(u_1, k_1)| \\
\leq & \max \{ |J'(u_0) - J'(u_1)|, |V'(u_0) - V'(u_1)|, |\phi'(u_0) - \phi'(u_1)| \} \\
\leq & \max \{D_J, D_V, D_\phi\} \|u_1 - u_0\| < \epsilon,
\end{align*}
\]

for any \( \psi \in \Psi \), as the Lipschitz bounds \( D_J, D_V \) and \( D_\phi \) are the same for any \( \psi \in \Ψ \).

From the above properties, it follows that the operator \( T \) satisfies the conditions of Schauder’s fixed point theorem (see Theorem 17.4 in Stokey, Lucas and Prescott 1989). Therefore there exists a \( \psi^* = (J^*, V^*, \phi^*) \in \Psi \) such that \( T\psi^* = \psi^* \). Since \( \psi^* \in \Psi \), we know that \( J \) and \( V \) are weakly increasing, and \( \phi \) is weakly decreasing. Moreover, since \( \psi^* = T\psi^* \) and since \( TJ \) and \( TV \) are strictly increasing and \( T\phi \) is strictly decreasing for any \( \psi = (J, V, \phi) \in \Psi \), it follows that \( J^* \) and \( V^* \) are strictly increasing and \( \phi^* \) is strictly decreasing.

Given \( \psi^* = (J^*, V^*, \phi^*) \), we can compute the law of motion for unemployment \( g_i^* \) under the assumption that the contract specifies the firing probability \( d_B^* \) and \( d_G^* \) given in (xx). Given \( \psi^* \) and \( g_i^* \), \( \phi^*(g_B(u)) < \phi^*(g_G(u)) \) for all \( u \in [0, 1] \) and, hence, the unique optimal contract between an individual firm and an individual worker specifies that the worker should exert effort and that the firing probabilities should be \( d_B^* \) and \( d_G^* \). Given \( \psi^* \) and \( g_i^* \), we can compute the gains from trade accruing to the firm and to the worker, \( F^* \) and \( G^* \), given an arbitrary wage. Given \( \psi^*, g_i^*, F^* \) and \( G^* \), we can compute the bargained wage \( w^* \). Clearly, the tuple \( (J^*, V^*, \phi^*, g_i^* d_i^*, w^*) \) constitutes an equilibrium. And, in this equilibrium, firms find it optimal to coordinate on firing workers when the realization of the sunspot is \( B \). We have therefore established the following theorem.
Theorem 2: Under conditions (4.4), (4.5) and (4.6), there exists a Recursive Equilibrium \((J^*, V^*, \phi^*, g_i^*, d_t^*, w^*)\) in which the unique optimal contract prescribes the firing probabilities \(d_B^t(u') > d_G^t(u')\).

5 Properties of Coordinated Equilibrium

In this section, we establish some of the central features of an equilibrium in which firms coordinate on firing their workers when the realization of the sunspot is \(B\). Figure 1 plots the unemployment rate at bargaining stage of period \(t + 1\) as a function of the unemployment rate at the bargaining stage of period \(t\) and given that the realization of the sunspot in period \(t + 1\) is \(B\) (blue solid line) or \(G\) (green dashed line). The law of motion for unemployment in the state \(G\), \(g_G(u)\), is such that \(g_G(0) = 0\) and \(g_G(u) = g_G(u_1) > 0\), and \(g_G(u_1) - g_G(u_0) \in (0, u_1 - u_0)\) for all \(u_0, u_1 \in [0, 1]\) with \(u_1 > u_0\). Since \(g_G(0) = 0\) and \(g_G'(u) < 1\), it follows that \(g_G(u) = u\) for \(u = u_G^* = 0\) and \(g_G(u) < u\) for all \(u \in (u_G^*, 1]\). Hence, as long as the realization of the sunspot is \(G\), the unemployment rate falls and converges towards the conditional steady-state \(u_G^* = 0\).

The law of motion for unemployment in the state \(B\), \(g_B(u)\), is such that \(g_B(0) > 0\), \(g_B(1) = g_G(1)\) and \(g_B(u) > g_G(u)\) for all \(u \in [0, 1]\). Moreover, \(g_B(u_1) - g_B(u_0)\) is strictly positive and strictly smaller than \((u_1 - u_0)(1 + \varphi_u)\) for all \(u_0, u_1 \in [0, 1]\) with \(u_1 > u_0\). Since \(g_B(0) > 0\) and \(g_B(1) = g_G(1) \leq 1\), it follows that there exists a unique \(u_B^* \in (0, 1]\) such that \(g_B(u) = u\) for \(u = u_B^*\) and \(g_B(u) > u\) for all \(u \in [0, u_B^*]\). Hence, as long as the realization of the sunspot is \(B\) and the initial unemployment rate is below \(u_B^*\), the unemployment rate increases and converges to the conditional steady-state

\[
u_B^* = \left\{ \frac{1 - q_1}{q_1 - q_0} \frac{c/\beta}{V(u_B^*)} \right\} \left[ \frac{1 - q_1}{q_1 - q_0} \frac{c/\beta}{V(u_B^*)} + \varphi(J(u_B^*), u_B^*) \right]. \tag{5.1}\]

As the realization of the sunspot varies over time, the unemployment rate will fluctuate, falling towards \(u_G^*\) when the realization of the sunspot is \(G\), and increasing towards \(u_B^*\) when the realization of the sunspot is \(B\). First, notice that these fluctuations in unemployment can be very large, as \(u_B^*\) can be much greater than \(u_G^*\). Second, notice that these fluctuations in unemployment are uncorrelated with fluctuations in labor productivity, which remains constant at the level \(y = q_1 y_H + (1 - q_1) y_L\). The features of the model are also distinctive features of the US labor market. First, the US labor market experiences large fluctuations in unemployment (see Figure 2). For instance, from October 1990 to June 1992, the unemployment rate increased from 5.9 to 7.8%. Similarly, from April 2008 to October 2009, the unemployment rate increased from 5.0 to 10.0%. Second, US unemployment fluctuations are only weakly correlated with labor productivity (see Figure 2). Indeed, over the period
1951-2012, the correlation between unemployment and the cyclical component of labor productivity is -0.25. Over the period 1990-2012, the correlation between unemployment and the cyclical component of labor productivity is 0.03.

The fact that our model can generate large unemployment fluctuations that are uncorrelated with labor productivity is worthy of an additional comment. The textbook search-theoretic model of unemployment cannot generate large fluctuations in unemployment in response to the fluctuations in labor productivity of the magnitude observed in the data (see, e.g., Shimer 2005). The literature has advanced several theories that can generate larger unemployment fluctuations in response to productivity shocks, including wage rigidities (Hall 2005, Kennan 2010, Menzio 2005), large number of marginal firm-worker matches (Menzio and Shi 2011), and small profit margins (Hagedorn and Manovskii 2010). However, all of these theories imply a nearly perfect positive correlation between unemployment and labor productivity.

The left panel in Figure 3 plots the probability that an employed worker becomes unemployed (henceforth, the EU rate) during the separation stage of period $t+1$ as a function of the unemployment rate at the bargaining stage of period $t$ and given that the realization of the sunspot in period $t+1$ is $B$ (blue solid line) or $G$ (green dashed line). When the realization of the sunspot is $G$, the EU rate is zero. When the realization of the sunspot is $B$, the EU rate is strictly positive and equal to $(1-q_1)d_B(g_B(u))$. The EU rate is decreasing.
in unemployment, as \( g_B(u) \) is strictly increasing in \( u \) and the firing probability \( d_B(u') \) are strictly increasing in \( u' \).

The right panel in Figure 3 plots the probability that an unemployed worker becomes employed (henceforth, the UE rate) during the matching stage of period \( t + 1 \) as a function of the unemployment rate at the bargaining stage of period \( t \) and given that the realization of the sunspot in period \( t + 1 \) is \( B \) (blue solid line) or \( G \) (green dashed line). When the realization of the sunspot is \( G \), the UE rate in is given by \( \varphi(J(g_G(u)), u) \). This rate is positive and is strictly decreasing in unemployment, as \( \varphi(J(g_G(u)), u) \) is strictly decreasing in \( u \) by virtue of condition (xx). When the realization of the sunspot is \( B \), the UE rate is given by \( \varphi(J(g_B(u)), u) \). This rate is also positive and strictly decreasing in unemployment. For any given \( u \), the UE rate is higher when the realization of the sunspot is \( B \) than when it is \( G \), as \( \varphi(J, u) \) is strictly decreasing in \( J \), \( J(u) \) is strictly increasing in \( u \) and \( g_B(u) \) is strictly greater than \( g_G(u) \). However, as long as the derivative of \( \varphi(J, u) \) with respect to \( J \) is small relative to the derivative with respect to \( u \), the distance between the UE rates for different realizations of the sunspot is small relative to the slope of the UE rate with respect to unemployment.

Figure 4 considers an economy that—in period \( t = 0 \)—has some unemployment rate \( u_0 < u_B^* \) and a realization of the sunspot \( G \). Suppose that in period \( t = 1 \) the realization of the sunspot switches from \( G \) to \( B \). On impact, the EU rate increases and the UE
rate will slightly increase. As long as the realization of the sunspot remains equal to \( B \), unemployment keeps increasing towards \( u^*_B \), the EU rate falls but always remains above its pre-shock level, and the UE rate declines and eventually falls below its pre-shock level. When the realization of the sunspot switches back to \( G \), unemployment starts falling towards \( u^*_G \), the EU rate falls immediately to its pre-recession level, while the UE rate slightly falls on impact and then starts recovering. Overall, the EU rate leads unemployment, while the UE rate tends to track unemployment. Intuitively, according to our theory, a period where the realization of the sunspot is \( B \), leads to an increases in the EU probability. This drives the unemployment rate up and, in turn, the increase in the unemployment rate drives the UE probability down. When the realization of the sunspot switches back to \( G \), the EU probability falls. This drives the unemployment rate down and, in turn, the decline in the unemployment rate drives up the UE probability.

The behavior of the US labor market is consistent with the view that the EU rate leads the unemployment rate and that the UE rate closely follows the unemployment rate. In fact, Fujita and Ramey (2009) show that the EU rate leads the unemployment rate, in the sense that the correlation between the unemployment rate in quarter \( t \) and the EU rate in quarter \( t + i \) is highest (in absolute value) for \( i = -1 \). Similarly, Fujita and Ramey (2009) show that the UE rate tracks the unemployment rate, in the sense that the correlation between the unemployment rate in quarter \( t \) and the EU rate in quarter \( t + i \) is highest (in absolute value) for \( i = 0 \).

The last implication of our theory that we want to discuss is that unemployment tends to increase in recessions (times when the realization of the sunspot is \( B \)) faster than it decreases in recoveries (times when the realization of the sunspot is \( G \)). In fact, during a
During a recovery, the decline in unemployment is given by
\[ u - g_G(u) = u' \varphi(J(g_G(u)), u). \]  
(5.3)

At \( u = 0 \), \( g_B(u) - u > 0 \) and \( u - g_G(u) = 0 \). By continuity, there exists a \( \hat{u} > 0 \) such that \( g_B(u) - u > u - g_G(u) \) for all \( u \in [0, \hat{u}] \). Hence, for any \( u \in [0, \hat{u}] \), unemployment increases more in a recession than it declines in a recovery. Also this implication of the theory is consistent with the data. Indeed, the reader can easily see from Figure 2 that the fluctuations in the unemployment rate are highly asymmetric. Recessions are characterized by a fast increase in unemployment and recoveries are characterized by a much slower decline in unemployment.

6 Conclusions
References


