Online Appendix for
“Agency Models with Frequent Actions: A Quadratic Approximation Method”
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B The HJB Equation

We start with a Lemma that establishes some basic properties of the solution of the HJB equation.

Lemma 17 Suppose that $\Theta(\bar{a}, \bar{h} \geq \theta > 0$ for all $\bar{a} > 0$.

(i) For any initial conditions $F(w)$ and $F'(w)$ the HJB equation (3) has a unique solution $F$ in any interval $[w, \bar{w}] \subset \mathbb{R}$.

(ii) $F$ is twice continuously differentiable and $(F, F')$ depends continuously on the initial conditions.

(iii) $F'$ is monotone with respect to $F'(w)$. That is, if $F_1$ and $F_2$ are two solutions of the HJB equation in an interval $[w, \bar{w}] \subset \mathbb{R}$ with $F_1(w) = F_2(w)$ and $F'_1(w) > F'_2(w)$, then $F'_1(w) > F'_2(w)$ (and hence $F_1(w) > F_2(w)$) for all $w > \bar{w}$.

Proof. See Sannikov [2008].

Note that the definition of $\Theta_\zeta$ for any $\zeta > 0$ guarantees that $\Theta_\zeta \geq \zeta > 0$. Lemma 17 thus implies that for any $\zeta > 0$ the HJB equation (6) with the boundary conditions (4) and (5) has a unique solution $F_\zeta$. Moreover, the uniqueness, continuity and monotonicity in the initial slope suggest the natural procedure for computing $F_\zeta$.

The following Lemma is crucial for the proof of the Theorems.

Lemma 18 Suppose that $\Theta(\bar{a}, \bar{h} \geq \theta > 0$ for all $\bar{a} > 0$. The solution $F$ of the HJB equation (3) with the boundary conditions (4) and (5) is strictly concave.

Proof. See Sannikov [2008].

The following “single crossing” Lemma will be used in the proof of Proposition 2.

Lemma 19 Consider two functions $\Theta \geq \Theta \geq 0$, and suppose that $F^{\Theta}$ and $F^{\Theta'}$ solve the corresponding HJB equations (3) with $F^{\Theta''} \leq 0$.

(i) If for some $w$, $F^{\Theta}(w) = F^{\Theta'}(w)$ and $F^{\Theta}(w') > F^{\Theta'}(w')$ in a right neighborhood of $w$, then $F^{\Theta}(w') > F^{\Theta'}(w')$ for all $w' > w$.

(ii) Assume $\Theta > \Theta$. If for some $w$, $F^{\Theta}(w) = F^{\Theta'}(w)$ and $F^{\Theta'}(w) \geq F^{\Theta'}(w)$, then $F^{\Theta'}(w') > F^{\Theta'}(w')$ for all $w' > w$. 

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Note that the precondition of part (i) is implied by (but is not equivalent to) $F^\Theta(w) = F^\Theta(w)$ and $F^\Theta(w) > F^\Theta(w)$.

**Proof.** We prove only part (i) (the proof of part (ii) is analogous). First, by assumption, $F^\Theta(w') > F^\Theta(w')$ for all $w' > w$ sufficiently close to $w$. Suppose now that there exists $w' > w$ with $F^\Theta(w') \leq F^\Theta(w')$ - we now assume that $w'$ is the smallest with this property. Since $F^\Theta >_{(w,w')} F^\Theta$, we have that $F^\Theta(w') > F^\Theta(w')$. Therefore, it must be the case that $F^\Theta(w') > F^\Theta(w')$: otherwise, since $F^\Theta(w') \leq 0$ and $\Theta \geq \Theta$, every policy $(\tilde{a}, \tilde{h}, c)$ would yield a weakly higher value of the right-hand side of HJB equation (3) for $F^\Theta(w')$ than for $F^\Theta(w')$. But $F^\Theta(w') > F^\Theta(w')$ implies that $F^\Theta(w') < F^\Theta(w''')$ for $w''$ in a left neighborhood of $w'$, contradicting the minimality of $w'$. ■

### B.1 Proof of Proposition 2, part (i)

For any $\zeta > 0$, let $F^\zeta$ and $F^\zeta$ be as in Theorem 1. By Lemma 17, $F^\zeta$ and $F^\zeta$ exist. Let us show that $F^\zeta(w) \leq F^\zeta(w)$ for all $w \in [0, w_{sp, \zeta}]$. Part (i) of Lemma 19 implies that there is no point $w \in [0, w_{sp, \zeta}]$ such that $F^\zeta(w) = F^\zeta(w)$ and $F^\zeta(w') > F^\zeta(w')$ in a right neighborhood of $w$: otherwise $F^\zeta(w_{sp, \zeta}) < F^\zeta(w_{sp, \zeta}) = F(w_{sp, \zeta})$, contradicting $F^\zeta \geq F$. The result attains since $F^\zeta(0) = F^\zeta(0)$.

The result also implies monotonicity in $\zeta$: for $\zeta \leq \zeta F^\zeta(w) \leq F^\zeta(w)$ for all $w \in [0, w_{sp, \zeta}]$ and $F^\zeta(w) \leq F^\zeta(w)$ for all $w \in [0, w_{sp, \zeta}]$. Altogether, we obtain $F^\zeta(w) \leq F^\zeta(w)$ for all $w \in [0, w_{sp}]$ by taking pointwise limits as $\zeta \downarrow 0$. This establishes the proof.

The following Corollary establishes the last missing part of the proof of Theorem 1.

**Corollary 3** For any $\Theta$ the Function $F$ and $w_{sp}$ in Theorem 1 exist.

### B.2 Proof of Proposition 1

We will use the following Lemma.

**Lemma 20**

(i) Suppose that $\Theta(\bar{a}, \bar{h}) \geq \Theta > 0$ for $\bar{a} > 0$. Then for all $\zeta > 0$ the $F_\zeta$ as in Theorem 1 solves equation (6) with an additional constraint $\bar{a} > 0$.

(ii) Suppose (Cont) holds. Then for any $[\underline{w}, \overline{w}] \subset (0, w_{sp})$ there exists $\gamma > 0$ such that for all sufficiently small $\zeta$, the $F_\zeta$ as in Theorem 1 solves equation (6) on $[\underline{w}, \overline{w}]$ with an additional constraint $\bar{a} \geq \gamma$.

**Proof.** For any $\lambda \in [F'(w_{sp}), \infty)$ let $H_\lambda$ be the linear function tangent to the retirement curve $\{(w, F(w)) : w \in [0, w_{sp}]\}$ with the slope $\lambda$ (if $\lambda \geq F'(0)$, $H_\lambda(w) = \lambda w$).

(i) Fix any $\zeta > 0$. Since $F_\zeta$ and $\overline{F}$ are concave and $F_\zeta \geq \overline{F}$, for any $w \in [0, w_{sp, \zeta}]$ we have $F_\zeta(w) - H_{\underline{F}_\zeta}(w) \geq 0$. On the other hand, for any $\zeta$ and $w \in [0, w_{sp}]$, if
we restrict the policy on the right hand side of equation (6) to satisfy $\bar{a} = 0$, we have
\[
\max\{c + F'_u(w)(w - u(c)) + \frac{1}{2} F''_u(w)r\zeta\} < \max\{c + F'_u(w)(w - u(c))\} = \bar{F}'(w') + F'_u(w')(w - w') = H_{F'_u(w)}(w),
\]
where $w'$ is such that $F'_u(w') = F'_u(w)$. Hence, choosing $\bar{a} = 0$ is never optimal and without any loss the additional constraint $\bar{a} > 0$ can be included in (6) to compute $F'_u(w)$.

(ii) We may assume $w_{sp} > 0$. Note also that for any $\zeta > 0$ and $F'_u$ as in Theorem 1 we have
\[
\inf_{a,w'} \left\{ \frac{F'_u(w')}{w} \right\} \leq F'_u(w) \leq \frac{F'_u(w)}{w},
\]
for all $w \in [w, \bar{w}]$. We will establish that there is $\alpha > 0$ such that for any $\zeta$ and $w \in [w, \bar{w}]$, $F'_u(w) - H_{F'_u(w)}(w) \geq \alpha$. If not, then let $\{w_n\}, \{w'_n\}, \{\zeta_n\}$ and $\{\alpha_n\}$ with $w_n \in [w, \bar{w}]$, $w'_n \leq w_{sp}$, $\zeta_n \downarrow 0$, $\alpha_n \downarrow 0$ be such that $F'_u(w_n) - H_{F'_u(w_n)}(w_n) \leq \alpha_n$ (where $w_n$ is such that $F'_u(w_n) = F'_u(w_n)$). We consider three cases.

(Case 1) Suppose that for some $\delta > 0$ and all $n$, $w'_n \in [\delta, w_{sp} - \delta]$. The concavity of $F'_u$ and $\bar{F}$ imply that $F'_u(w_n) - H_{F'_u(w_n)}(w_n) \geq F'_u(w'_n) - H_{F'_u(w'_n)}(w'_n) = F'_u(w'_n) - \bar{F}(w'_n)$. But, since $F'_u$ is increasing as $\zeta_n \downarrow 0$ (Proposition 2, part (i)), $F'_u(w'_n) - \bar{F}(w'_n) \geq \inf_{w \in [\delta, w_{sp} - \delta]} F'_u(w) - \bar{F}(w) > 0$, a contradiction.

(Case 2) If $w'_n \downarrow 0$ (we might assume so by choosing a subsequence), then we would have $F'_u(w_n) \rightarrow H_{F'_u(w_n)}(w_n) \rightarrow \bar{F}(0) \times w_n$. By concavity of all $F'_u$, this would imply that, first, $F'_u(w) \rightarrow \bar{F}'(0) \times w$ for all $w \in [0, w_n]$, and second, that there is a sequence $\{w'_n\}, w'_n \in [0, w_n]$, such that $F'_u(w'_n) \rightarrow \bar{F}'(0)$ and $F'_u(w''_n) \rightarrow 0$. But then
\[
F'_u(w'_n) \rightarrow \max_{a,c} \{ (c - F'(0)(w'_n + h(a)) - u(c)) \} = \max_{a} \{ a + \bar{F}'(0)(w''_n + h(a)) \} > \bar{F}'(0)w''_n,
\]
where the equality follows from the fact that $\bar{F}'(0) = \frac{1}{w'(0)}$ and strict concavity of $u$, while the inequality follows from $h'_n(0) < u'(0)$. This establishes the required contradiction.

(Case 3) If $w'_n \uparrow w_{sp}$, we derive the contradiction in the analogous way as in case 2.

We have established that for all $\zeta$ and $w \in [w, \bar{w}]$, $F'_u(w) - H_{F'_u(w)}(w) \geq \alpha > 0$. On the other hand, for any $\zeta$ and $w \in [w, \bar{w}]$, if we restrict the policy on the right hand side of equation (6) to satisfy $\bar{a} = 0$, we have $\max_{c} \{-c + F'_u(w)(w - u(c)) + \frac{1}{2} F''_u(w)r\zeta\} \leq H_{F'_u(w)}(w) \leq F'_u(w) - \alpha$. Since $F'_u$ are uniformly bounded on $[w, \bar{w}]$ and $\bar{h} \leq \frac{\bar{c}}{\bar{r}} h(A)$, we see that the policy for $F'_u(w)$, for any $\zeta$ and $w \in [w, \bar{w}]$, satisfies the additional constraint $\bar{a} \geq \gamma$ for appropriate $\gamma > 0$. ■

We are now ready to prove Proposition 1.

(i) Given Lemma 20, part (i), all the functions $F'_u$ for $\zeta < \theta$ are the same, and solve the equation (3). The Proposition then follows from Theorem 1.

(ii) Choose any $[w, \bar{w}] \subset (0, w_{sp})$. Lemma 20, part (ii), guarantees that for sufficiently small $\zeta$ all $F'_u$ satisfy the constraint $\bar{a} \geq \gamma$ on $[w, \bar{w}]$, for some $\gamma > 0$. Therefore, for
sufficiently small $\zeta$ all $F_\zeta$ satisfy on $[w, \bar{w}]$:

$$F''(w) = \inf_{a \geq \gamma, h, c} \left\{ \frac{F(w) - (\bar{a} - c) - F'(w)(w + h - u(c))}{r\Theta(\bar{a}, h)/2} \right\},$$

with the right-hand side Lipschitz continuous in $(w, F(w), F'(w))$, since $\Theta \geq \delta(\gamma) > 0$ for $\bar{a} \geq \gamma$.

Part (i) of Proposition 2 guarantees that $F_\zeta$ converge in the supremum norm as $\zeta \downarrow 0$ to a function $F$. Since $F'_\zeta$ are uniformly bounded on $[w, \bar{w}]$, it follows that all $F''_\zeta$ and $F'_\zeta$ are Lipschitz continuous with the same Lipschitz constant, and so $F'_\zeta$ converge to $F'$ not only in $L^1$ but in the supremum norm, by the Arzela-Ascoli Theorem. Uniform Lipschitz continuity guarantees also that $F = \frac{d}{dw}F$, that $F'' := \lim_{\zeta \downarrow 0} F''_\zeta$ exists and $F$ satisfies the above equation (all on $[w, \bar{w}]$). Since the set $[w, \bar{w}]$ is arbitrary, this proves that $F$ solves (3) in $(0, w_{sp})$.

**B.3 Proof of Proposition 2, part (ii)**

Part (ii) of Proposition 1 shows that $F_\Theta$ and $F'_\Theta$ satisfy the HJB equation (3). Part (ii) of Lemma 20 guarantees that for $F_\Theta$ and $F'_\Theta$ restricted to $[w, \bar{w}] \subset (0, w_{sp}, \Theta)$ as well as the domains of $\Theta$ and $\Theta$ restricted to $\{(\bar{a}, h) : \bar{a} \geq \gamma\}$, the preconditions for part (ii) of Lemma 19 are satisfied, with $\theta = \delta(\gamma)$ for $\gamma$ as in Lemma 20. Moreover, part (i) of the Proposition implies that $F_\Theta \geq F'_\Theta$. Therefore, if $F_\Theta(w) = F'_\Theta(w)$ for some $w \in (0, w_{sp})$, it must be that $F_\Theta(w) = F'_\Theta(w)$, and so $F''_\Theta(w') < F''_\Theta(w')$ for $w' > w$, contradicting $F_\Theta \geq F$. This completes the proof.

**B.4 Proof of Proposition 3**

The proof follows from the following Lemma.

**Lemma 21** Suppose that $\Theta \equiv 0$. For any $\delta > 0$ and sufficiently small $\zeta$ the solution $F_\zeta$ of the HJB equation (6) with initial conditions

$$F_\zeta(\bar{w}) = \bar{F}(\bar{w}) - \delta, \quad F'_\zeta(\bar{w}) = \bar{F}'(\bar{w})$$

for some $\bar{w} \in [0, w_{sp}]$ satisfies

$$F''_\zeta \leq_{[0, w_{sp}]} -\frac{2\delta}{\zeta}.$$

**Proof.** For any $\lambda \in [\bar{F}'(w_{sp}), \infty)$ let $G_\lambda$ be the linear function tangent to the first-best frontier $\{(w, \bar{F}(w)) : w \in [0, w_{sp}]\}$ with the slope $\lambda$. We will show that if for an arbitrary $w \in [0, w_{sp}]$

$$G_{F_\zeta}(w) - F_\zeta(w) \geq \delta,$$  

(31)
then \( F''_\xi(w) \leq -\frac{2\delta}{\zeta} \). Note that then as long as \(-\frac{2\delta}{\zeta} \leq \min_{w \in [0, w_{sp}]} F''(w)\) the above condition will be satisfied over the whole interval \([0, \bar{w}_{sp}]\), which will establish the Lemma.

Since \( \Theta \equiv 0 \), the HJB equation (6) takes the form

\[
F''_\xi(w) = \frac{2}{\zeta} \min_{a, h, c} \left\{ F_\xi(w) - (a - c) - F'_\xi(w)(w + h - u(c)) \right\}.
\]  

(32)

Let \( w' \) be such that \( F'_\xi(w) = F'(w') \). For the policy \( (a(w'), c(w')) \) in the problem (1) at \( w' \) we have:

\[
F_\xi(w) - (a(w') - c(w')) - F'_\xi(w)(w + h(a(w')) - u(c(w')))
\]

\[
F'(w') - (a(w') - c(w')) - F'(w')(w' + h(a(w')) - u(c(w')))
\]

\[
+ [F_\xi(w) - F'(w') + F'_\xi(w)(w' - w)] = [F_\xi(w) - F'(w') + F'_\xi(w)(w' - w)] \leq -\delta,
\]

where the last equality follows from (1), while the last inequality follows from (31). Since \((a(w'), h(a(w')), c(w'))\) is an available policy in the problem (32), this establishes that

\[
F''_\xi(w) \leq -\frac{2\delta}{\zeta}.
\]

Given the Lemma, for any \( \delta > 0 \) and sufficiently small \( \zeta > 0 \) the solution \( F_\xi \) of the HJB equation (6) with initial conditions \( F_\xi(\bar{w}) = F(\bar{w}) - \delta, F'_\xi(\bar{w}) = F'(\bar{w}) \) with \( \bar{w} \in [\delta, \bar{w}_{sp}] \) will satisfy \( F_\xi(w) = F(w) \) and \( F'_\xi(w) = F'(w) \) for some \( 0 < w < \bar{w} < \bar{w}_{sp} \).

This together with Proposition 7 and part (ii) of Lemma 6 establishes the proof of the Proposition.

C Proof of Proposition 4

Fix period length \( \Delta > 0 \), densities \( g_X \) and \( \gamma_X \) satisfying (12) and any \( w_g, w_\gamma \in [0, \bar{w}] \).

Consider the problem of finding a contract \( \{c_n\} \) and action plans \( \{a_{g,n}\}, \{a_{\gamma,n}\} \) that maximize the sum of principal’s expected discounted revenues under noise densities \( g_X \) and \( \gamma_X \), such that \( \{c_n\}, \{a_{g,n}\} \) is incentive compatible under \( g_X \) and \( \{c_n\}, \{a_{\gamma,n}\} \) is incentive compatible under \( \gamma_X \), and they deliver expected discounted utilities \( w_g \) and \( w_\gamma \) to the agent. Let \( F_{g,\gamma}^{\Delta}(w_g, w_\gamma) \) be the value to the principal from the optimal contract:

\[
F_{g,\gamma}^{\Delta}(w_g, w_\gamma) = \sup \left\{ \Pi_g(\{c_n\}, \{a_{g,n}\}) + \Pi_\gamma(\{c_n\}, \{a_{\gamma,n}\}) \right\}
\]

\( \{a_{g,n}\} \) is IC for \( \{c_n\} \), \( U_g(\{c_n\}, \{a_{g,n}\}) = w \) under density \( g_X \),

\( \{a_{\gamma,n}\} \) is IC for \( \{c_n\} \), \( U_\gamma(\{c_n\}, \{a_{\gamma,n}\}) = w \) under density \( \gamma_X \)

To establish the Proposition we show that if \( w_g, w_\gamma \in (0, w_{sp}) \) then there is \( \delta > 0 \) such that for sufficiently small \( \Delta \) \( F_{g,\gamma}^{\Delta}(w_g, w_\gamma) + \delta \leq F(w_g) + F(w_\gamma) =: F_2(w_g, w_\gamma) \), where \( F \) is as in Theorem 1.
First, consider the following Bellman operator:

$$T_{g,\gamma}^\Delta f(w_g, w_\gamma) = \sup_{a_g, a_\gamma, c, W_g, W_\gamma} \Phi_g^\Delta(a_g, c, W_g; f) + \Phi_\gamma^\Delta(a_\gamma, c, W_\gamma; f)$$

s.t. $a_\phi \in A$, $c(y) \geq 0$ and $W_\phi(y) \in [0, \bar{u})$ \ \forall y$

$$w_\phi = E_\phi^\Delta \left[ \bar{r} \Delta[u(c(\Delta[x + a_\phi])) - h(a_\phi)] + e^{-r\Delta} W_\phi(\Delta[x + a_\phi]) \right] \quad (PK2)$$

$$a_\phi \in \arg \max_{\hat{a} \in A} E_\phi^\Delta \left[ \bar{r} \Delta[u(c(\Delta[x + \hat{a}])) - h(\hat{a})] + e^{-r\Delta} W_\phi(\Delta[x + \hat{a}]) \right] \quad (IC_2-AC)$$

where the supremum is taken over measurable functions and and $\Phi^\Delta_\phi(a, c, W; f)$ is as in (9), for $\phi \in \{g, \gamma\}$. The following is an analogue of Proposition 5:

**Proposition 8** $F_{g,\gamma}^\Delta$ is the largest fixed point $f$ of $T_{g,\gamma}^\Delta$ such that $f(w_g, w_\gamma) \leq F(w_g) + F(w_\gamma)$.

For a set of feasible policies $p = \{(a_g, a_\gamma, c, W_g, W_\gamma)\}_{(w_g, w_\gamma) \in [0, \bar{u})^2}$ for the Bellman operator $T_{g,\gamma}^\Delta$, let $T_{g,\gamma}^{\Delta,p}$ be the operator defined as $T_{g,\gamma}^{\Delta,p} f(w) = \Phi_g^\Delta(a_g, c, W_g; f) + \Phi_\gamma^\Delta(a_\gamma, c, W_\gamma; f)$, and let $F_{g,\gamma}^{\Delta,p}$ be the value achieved by the policies $p$. Note that $F_{g,\gamma}^{\Delta,p}$ is a fixed point of $T_{g,\gamma}^{\Delta,p}$. Also, policies $p$ together with an initial point $(w_g, w_\gamma) = (w_{g,0}^p, w_{\gamma,0}^p)$ determine a stochastic process $\{(w_{g,n}, w_{\gamma,n})\}$ of continuation values.

For the proof of the Proposition we use the following five claims. Claim 1 is related to Lemma 6. It shows that for a fixed set of policies $p$ for the Bellman operator $T_{g,\gamma}^\Delta$, how far the value of the contract built up recursively from those policies falls short of $F_2$ ($F_2 - F_{g,\gamma}^{\Delta,p}$) can be expressed as a discounted expected sum of how far each policy applied to $F_2$ falls short of $F_2$ ($F_2 - T_{g,\gamma}^{\Delta,p}F_2$).

The idea behind the construction in the remaining four claims is as follows. For any $\varepsilon > 0$ consider the set $S_\varepsilon = \{(w_g, w_\gamma) \in [\varepsilon, w_{\text{sp}} - \varepsilon] : |w_g - w_\gamma| > \varepsilon, \max \{w_g, w_\gamma\} > w_0 + \varepsilon\}$, where $w_0$ is such that $F''(w_0) = F'(0) = -\frac{1}{w(0)}$. Claim 2 shows that once the two continuation values are in the set, $F_2 - T_{g,\gamma}^{\Delta,p}F_2$ must be negative: The reason is that to achieve $F_2 (w_g, w_\gamma) = F(w_g) + F(w_\gamma)$ the wages paid in the separate two optimal policies for each continuation value must be different (such that $-1/u'(c_g) = F'(w_g)$, and $-1/u'(c_f) = F'(w_f)$), whereas $T_{g,\gamma}^{\Delta,p}$ restricts the wage to be the same.

Claim 3 shows that if $F_2 - T_{g,\gamma}^{\Delta,p}F_2$ is to remain small, it must be that the variances of continuation values $W_g, W_f$ and $W_g - W_f$ must be bounded away from zero, and not too big. This follows from the results in the paper: for the policy $p$ to fare well, the continuation values for each noise must be approximately linear in likelihood ratio. Also, since the likelihood ratios are linearly independent by assumption, $W_g - W_f$ can't be too small. Using Claim 3, Claim 4 shows that under policies $p$ once the process of continuation values $\{w_g, w_\gamma\}$ enters set $S_\varepsilon$, it must stay there for a while with nonnegligible probability; Claim 5 shows that starting at any interior point of continuation values the process enters $S_\varepsilon$ in finite time with nonnegligible probabilities. Those results, together with Claim 2 establish the Proposition.

Fix a set of policies $p$ for the Bellman operator $T_{g,\gamma}^\Delta$. 
Claim 1 Consider function $F : [0, \bar{u})^2 \to \mathbb{R}$ and $(w_{g,0}^p, w_{\gamma,0}^p) \in [0, \bar{u})^2$. Then for any $N \in \mathbb{N}$

$$F_2(w_g, w_\gamma) - F_{g,\gamma}^\Delta (w_g, w_\gamma) = \mathbb{E}_{g,\gamma} \left[ \sum_{n=0}^{N} e^{-rn\Delta} (F_2(w_{g,n}^p, w_{\gamma,n}^p) - T_{g,\gamma}^\Delta F_2(w_{g,n}^p, w_{\gamma,n}^p)) + e^{-r(N+1)\Delta} (F_2(w_{g,N+1}^p, w_{\gamma,N+1}^p) - F_{g,\gamma}^\Delta (w_{g,N+1}^p, w_{\gamma,N+1}^p)) \right].$$

Proof. For any $(w_g, w_\gamma) \in [0, \bar{u})^2$ we have

$$F_2(w_g, w_\gamma) - F_{g,\gamma}^\Delta (w_g, w_\gamma) = F_2(w_g, w_\gamma) - T_{g,\gamma}^\Delta F_{g,\gamma}^\Delta (w_g, w_\gamma) =$$

$$= \mathbb{E}_{g,\gamma} \left[ F_2(w_g, w_\gamma) - T_{g,\gamma}^\Delta F_2(w_g, w_\gamma) + T_{g,\gamma}^\Delta F_2(w_g, w_\gamma) - T_{g,\gamma}^\Delta F_{g,\gamma}^\Delta (w_g, w_\gamma) \right] =$$

and using the equality recursively yields the proof. ■

Claim 2 Fix $\varepsilon > 0$ and $(w_g, w_\gamma) \in S_\varepsilon$. Then there is $\delta_1$ such that for sufficiently small $\Delta > 0$:

$$F_2(w_g, w_\gamma) - T_{g,\gamma}^\Delta F_2(w_g, w_\gamma) > \delta_1 \Delta.$$ 

Proof. In analogy to $T_g^\Delta$ we also define a simplified “quadratic” operator $T_{g,\gamma}^\Delta$:

$$T_{g,\gamma}^\Delta f(w_g, w_\gamma) = \sup_{a_g, a_\gamma, c, W_g, f, w_g} \Phi_g^\Delta(a_g, c, W_g; f, w_g) + \Phi_\gamma^\Delta(a_\gamma, c, W_\gamma; f, w_\gamma)$$

s.t. $a_\phi(z) \in \mathcal{A}$, $c \geq 0$, and $W_\phi(y) \in \mathbb{R}$ $\forall y$

$$w_\phi = \mathbb{E}_\phi \left[ \tilde{r} \Delta [u(c(\Delta x + a_\phi))] - h(a_\phi)] + e^{-r\Delta} W_\phi(\Delta x + a_\phi)] \right] \quad (PK_2 q)$$

$$\tilde{r} h'(a_\phi) = -\frac{e^{-r\Delta}}{\Delta} \int_{\mathbb{R}} W(\Delta x) \phi_X^\Delta(x)dx \quad (FOC_2 q-AC)$$

where the supremum is taken over measurable functions and $\Phi_\phi^\Delta(a, c, W; f, w_\phi)$ is defined in (16), for $\phi \in \{g, \gamma\}$. Using analogues to Lemmas 9 and 12 we establish:

$$|T_{g,\gamma}^\Delta F_2 - T_{g,\gamma}^\Delta F_2|_{[0, \bar{u})^2} = o(\Delta).$$

Fix $\varepsilon > 0$ and $(w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2$ such that $|w_g - w_\gamma| \geq \varepsilon$. In view of the above bound, it is sufficient to establish that $F_2(w_g, w_\gamma) - T_{g,\gamma}^\Delta F_2(w_g, w_\gamma) > \delta_1 \Delta$, and so, due to Proposition 7 and Lemmas 9 and 12, it is sufficient to show that

$$T_g^\Delta F(w_g) + T_{g,\gamma}^\Delta F(w_\gamma) - T_{g,\gamma}^\Delta F_2(w_g, w_\gamma) > \delta_1 \Delta,$$

where $T_g^\Delta$ and $T_{g,\gamma}^\Delta$ stand for operator $T^\Delta$ under the respective noise densities.
We have
\[ T_{\phi}^\Delta F(w_\phi) = \sup_c -\bar{\tau} \Delta \{ c + F'(w_\phi) u(c) \} + \sup_{a,W} \Psi_{\phi}^\Delta (a,W;F,w_\phi), \]
\[ T_{g,\gamma}^\Delta F_2(w_g,w_\gamma) = \sup_c -\bar{\tau} \Delta \{ 2c + F'(w_g) u(c) + F'(w_\gamma) u(c) \} + \sup_{a_g,w_g} \Psi_{g}^\Delta (a_g,W_g;F,w_g) + \sup_{a_\gamma,w_\gamma} \Psi_{g}^\Delta (a_\gamma,W_\gamma;F,w_\gamma), \]
where
\[ \Psi_{\phi}^\Delta (a,W;F,w_\phi) = e^{-\Delta r} F(w_\phi) + \bar{\tau} \Delta \{ a + F'(w_\phi)[w_\phi + h(a_\phi)] \} + e^{-\Delta r} \mathbb{E}_{\phi}^\Delta \left[ \frac{1}{2} F''(w_\phi) (W(\Delta x) - w_\phi)^2 \right], \]
\[ \phi \in \{ g, \gamma \}. \]
The proof follows from the fact that \( F'' \) is bounded away from 0 on \([0, w_{SP}]^2 \) and so \( |F'(w_g) - F'(w_\gamma)| > \varepsilon_1 \) for some \( \varepsilon_1 > 0 \), which implies that for some \( \delta_1: \)
\[ \sup_c \{ c + F'(w_g) u(c) \} + \sup_c \{ c + F'(w_\gamma) u(c) \} > \sup_c \{ 2c + F'(w_g) u(c) + F'(w_\gamma) u(c) \} + \delta_1. \]

**Claim 3** Fix \( \varepsilon > 0 \) and \((w_g,w_\gamma) \in [\varepsilon, w_{SP} - \varepsilon]^2 \). Then there is \( \delta_2 > 0 \) such that for sufficiently small \( \Delta \) and any feasible policy \((a_g,a_\gamma,c,W_g,W_\gamma)\) for \( T_{g,\gamma}^\Delta F_2(w_g,w_\gamma) \) if
\[ \Phi_g^\Delta (a_g,c,W_g,F_2) + \Phi_\gamma^\Delta (a_\gamma,c,W_\gamma,F_2) > F_2(w_g,w_\gamma) - \delta_2 \Delta \]
then
\[ \forall [W_g(\Delta(x_g + a_g))] \cup [W_\gamma(\Delta(x_\gamma + a_\gamma))] > \delta_2 \Delta, \]
\[ \forall g,\gamma [W_g(\Delta(x_g + a_g)) - W_\gamma(\Delta(x_\gamma + a_\gamma))] > \delta_2 \Delta. \]

On the other hand,
\[ F_2(w_g,w_\gamma) - \Phi_g^\Delta (a_g,c,W_g,F_2) + \Phi_\gamma^\Delta (a_\gamma,c,W_\gamma,F_2) > \delta_2 \left( \forall [W_g(\Delta(x_g + a_g))] \right) - \Delta \frac{(r'h(A))^2}{VLR(gx)} + \delta_2 \left( \forall [W_\gamma(\Delta(x_\gamma + a_\gamma))] \right) - \Delta \frac{(r'h(A))^2}{VLR(gx)} \]

**Proof.** Lemmas 20 part ii) and 7 imply that for certain \( \delta_2 > 0 \) and sufficiently small \( \Delta \) if \( \Phi_g^\Delta (a_g,c,W_g,F_2) + \Phi_\gamma^\Delta (a_\gamma,c,W_\gamma,F_2) > F_2(w_g,w_\gamma) - \delta_2 \Delta \), then \( a_g,a_\gamma > a > 0 \). But then Lemmas 7 and 1 imply that \( \forall [W_\phi(\Delta(x_\phi + a_\phi))] \approx \Delta \frac{(r'h(A))^2}{VLR(\phi X)}, \) for \( \phi \in \{ g, \gamma \}, \) which yields the first inequality. The same Lemmas imply that \( W_\phi(\Delta(x_\phi + a_\phi)) \approx \mathbb{E}_{\phi}^\Delta [W_\phi(\Delta(x_\phi + a_\phi))] + \sqrt{\Delta D_{g(x)} g'(x)} \) (in \( L_1 (\phi X) \)), for \( \phi \in \{ g, \gamma \} \), and so the second inequality follows from the linear independence of likelihood ratios (12). Finally, \( F'' \) bounded away from zero immediately implies the third inequality. 

Fix a set of policies \( p \) for the Bellman operator \( T_{g,\gamma}^\Delta, T > 0, \varepsilon_1 > \varepsilon_2 > 0. \)
Claim 4 Fix an initial point \((w_g, w_\gamma) \in S_\varepsilon\). Then there are \(\delta_3, T > 0\) such that for sufficiently small \(\Delta\)
\[
\mathbb{P}^{\Delta}_{g,\gamma} \left[ \sum_{n=0}^{T/\Delta} e^{-rn\Delta} (F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^\Delta_{g,\gamma} F_2(w^p_{g,n}, w^p_{\gamma,n})) \right] \leq \delta_3
\]
implies
\[
\mathbb{P}^{\Delta}_{g,\gamma} [(w^p_{g,n}, w^p_{\gamma,n}) \in S_{\varepsilon/2}, n = 0, \ldots, T/\Delta] > \delta_3.
\]

Proof. If the precondition is satisfied, then Claim 3 implies that, for \(\phi \in \{g, \gamma\},\)
\[
\forall \phi \left[ (w^p_{\phi,t} - w^p_{\phi,0}) \right] \leq T \left( \frac{r h'(A)^2}{V LR(g_\gamma)} \varepsilon / \delta' \right) =: C_{T,\delta}, \text{ for } t \leq T / \Delta,
\]
\[
\mathbb{E}^{\Delta}_{\phi} \left[ \left| w^p_{\phi,T/\Delta} - w^p_{\phi,0} \right| w^p_{\phi,0} \right] \leq C_{T,\delta}, \text{ for } t \leq T / \Delta,
\]
with \(C_{T,\delta} \to 0\) as \(T, \delta \to 0\). We also have
\[
\mathbb{E}^{\Delta}_{\phi} [(w^p_{\phi,t} - w^p_{\phi,t'})] w^p_{\phi,t'} \leq D_{T,\delta}, \text{ for } t' < t' \leq T / \Delta,
\]
with \(D_{T,\delta} \to 0\) as \(T, \delta \to 0\). It therefore follows that for \(\alpha = \frac{\varepsilon}{4} > 0\) and \(\tau\) the stopping time of reaching the set \([\alpha, \infty)\)
\[
\mathbb{P}^{\Delta}_{\phi} \left[ \max_{t \leq T / \Delta} \left| w^p_{\phi,t} - w^p_{\phi,0} \right| \geq \alpha \right]
= \mathbb{P}^{\Delta}_{\phi} \left[ \max_{t \leq T / \Delta} \left| w^p_{\phi,t} - w^p_{\phi,0} \right| \geq \alpha, w^p_{\phi,T/\Delta} - w^p_{\phi,0} \geq \alpha/2 \right] + \mathbb{P}^{\Delta}_{\phi} \left[ \max_{t \leq T / \Delta} \left| w^p_{\phi,t} - w^p_{\phi,0} \right| \geq \alpha, w^p_{\phi,T/\Delta} - w^p_{\phi,0} < \alpha/2 \right]
\leq \mathbb{P}^{\Delta}_{\phi} \left[ w^p_{\phi,T/\Delta} - w^p_{\phi,0} \geq \alpha/2 \right] + \mathbb{P}^{\Delta}_{\phi} \left[ w^p_{\phi,T/\Delta} - w^p_{\phi,\tau} < -\alpha/2 \right] \leq 2 \frac{C_{T,\delta}}{(\alpha/2 - D_{T,\delta})^2} \to 0,
\]
as \(T, \delta \to 0\). This establishes the proof. \(\blacksquare\)

Claim 5 Fix an initial point \((w_g, w_\gamma) \in [\varepsilon, w_{sp} - \varepsilon]^2\). Then there are \(\delta_4, T > 0\) such that for sufficiently small \(\Delta\)
\[
\mathbb{P}^{\Delta}_{g,\gamma} \left[ \sum_{n=0}^{T/\Delta} e^{-rn\Delta} (F_2(w^p_{g,n}, w^p_{\gamma,n}) - T^\Delta_{g,\gamma} F_2(w^p_{g,n}, w^p_{\gamma,n})) \right] \leq \delta_4
\]
implies
\[
\mathbb{P}^{\Delta}_{g,\gamma} [(w^p_{g,T/\Delta+1}, w^p_{\gamma,T/\Delta+1}) \in S_{\varepsilon}] > \delta_4.
\]
Proof. The proof is similar to the proof of the previous claim and so is omitted. ■

Given the claims, the rest of the proof is as follows. If \((w_g, w_\gamma) \in S_\epsilon\) then for the constants as in the claims

\[
F_2(w_g, w_\gamma) - F_{g,\gamma}^{\Delta,p}(w_g, w_\gamma) \geq \mathbb{E}_{g,\gamma}^{\Delta} \left[ \sum_{n=0}^{T/\Delta} e^{-r n \Delta} (F_2(w_{g,n}^p, w_{\gamma,n}^p) - T_{g,\gamma}^{\Delta,p} F_2(w_{g,n}^p, w_{\gamma,n}^p)) \right]
\]

\[
\geq \min \left\{ \delta_3, \frac{1 - e^{-r T}}{1 - e^{-r \Delta}} \delta_3 \delta_1 \right\},
\]

where the first inequality follows from Claim 1 and the second inequality follows from Claims 2 and 4.

If on the other hand \((w_g, w_\gamma) \in [\varepsilon_1, w_{sp} - \varepsilon]^2 \setminus S_\epsilon\) then

\[
F_2(w_g, w_\gamma) - F_{g,\gamma}^{\Delta,p}(w_g, w_\gamma) \geq \mathbb{E}_{g,\gamma}^{\Delta} \left[ \sum_{n=0}^{T/\Delta} e^{-r n \Delta} (F_2(w_{g,n}^p, w_{\gamma,n}^p) - T_{g,\gamma}^{\Delta,p} F_2(w_{g,n}^p, w_{\gamma,n}^p)) \right]
\]

\[
+ e^{-r(T+\Delta)} (F_2(w_{g,T/\Delta+1}^p, w_{\gamma,T/\Delta+1}^p) - T_{g,\gamma}^{\Delta,p} F_2(w_{g,T/\Delta+1}^p, w_{\gamma,T/\Delta+1}^p)) \right]
\]

\[
\geq \min \left\{ \delta_4, e^{-r(T+\Delta)} \delta_4 \min \left\{ \delta_3, \frac{1 - e^{-r(T+\Delta)}}{1 - e^{-r \Delta}} \delta_3 \delta_1 \right\} \right\},
\]

where the first inequality follows from Claim 1 and the second inequality follows from Claim 5 and the inequalities above. This establishes the proof of the Proposition.

We note that the proof can be extended beyond the pure hidden action case and \(VLR(g_X) = VLR(\gamma_X)\). As regards the equality of variances of likelihood ratios, this guaranteed that the limits of the values of contracts \(F_g\) and \(F_\gamma\) for two noise distributions are the same function \(F\) (Lemma 1). Because of that, as long as the continuation values \(w_g\) and \(w_\gamma\) are not the same the derivatives \(F'_g(w_g)\) and \(F'_\gamma(w_\gamma)\) differ as well, which is crucial for Claim 2. Dropping the assumption \(VLR(g_X) = VLR(\gamma_X)\) the proof would be analogous, yet the computation of the set of continuation values \((w_g, w_f)\) for which \(F'_g(w_g) \neq F'_\gamma(w_\gamma)\) would be cumbersome.

On the other hand, the assumption of pure hidden action models was also not crucial for the proof: For two different information structures the proof will work as long as, roughly, the optimal policies in the problem of minimizing variance of incentive transfers are sufficiently different (see Claim 3).