An Equilibrium Theory of Learning, Search and Wages[*]

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Abstract

We propose an equilibrium theory of learning from search in the labor market, which addresses the search behavior of workers, the creation of jobs, and the determination of wages as functions of labor market histories. In the model, each worker has incomplete information about his job-finding ability and learns about it from his search outcomes. The theory formalizes a notion akin to discouragement: over each uninterrupted unemployment spell, unemployed workers update their beliefs about their job-finding abilities downward and reduce their desired wages. One contribution of the paper is to integrate learning from search into an equilibrium framework. By inducing endogenous heterogeneity in workers' beliefs, learning from search provides a novel explanation for a set of related empirical observations about the labor market, including unemployment duration dependence and wage dispersion. Another contribution is to apply lattice-theoretic techniques to analyze learning from experience, which is useful because learning generates convex value functions and, in principle, multiple solutions to a worker's optimization problem.

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1. Introduction

Learning by workers with incomplete information about their own job-finding process is a central feature of job search. Intuitively, differences in search outcomes generate endogenous heterogeneity in workers' beliefs about their job-finding process, which leads to further differences in labor market outcomes. In this paper, we develop an equilibrium theory that captures this learning process in the labor market. The main assumption is that each worker has imprecise knowledge of his ability to form a productive match with an employer and, therefore, learns about his ability from search outcomes. We embed this assumption in a model of directed search to characterize an equilibrium that determines not only workers' search behavior, but also job creation and the wage distribution.

Although our analysis is theoretical, it is also motivated by a set of related empirical observations about the labor market, including the following. First, unemployment is associated with large and persistent losses in future wages, and the longer a worker's unemployment duration, the larger the wage loss the worker will experience after reemployment. This effect of unemployment duration is significant after controlling for observed and unobserved heterogeneity. Second, average job-finding probabilities fall with unemployment duration. The bulk of this negative duration dependence can be accounted for by heterogeneity among workers. Third, vacancies offering lower wages are filled less rapidly, even though workers who seek those jobs have longer unemployment durations and lower job-finding probabilities. Fourth, a worker's unemployment duration, as well as the reemployment wage, increases with the worker's pre-unemployment wage, after controlling for workers' observable characteristics. It is unlikely that these observations reflect variations in human capital alone, since workers' labor market histories have large and persistent effects on their future labor market outcomes after controlling for worker characteristics.[1]

Learning from search provides a novel explanation for the above observations. To formalize this explanation, we consider a labor market in which a worker has imperfect knowledge of his ability, which is either high or low permanently. High ability increases the probability that the worker is productive when matched at random with a job. We call this probability the worker's efficiency units in search. There is a continuum of submarkets, each of which is associated with a wage and tightness. The statistics of all submarkets are public information. Firms and workers choose which submarket to enter, understanding

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[1] See, for example, Addison and Portugal (1989) and Nickell et al. (2002) for evidence on the relationship between unemployment duration and wages, Machin and Manning (1999) for evidence on unemployment duration dependence, and Barron et al. (1985) and Holzer et al. (1991) for evidence on the relationship between wage offers and the duration of vacancies.

that a submarket with a higher wage has relatively fewer vacancies. We refer to this choice as the agent's search decision and the wage in the chosen submarket as the worker's desired wage. In each submarket, the number of productive matches is a function of the number of vacancies and total efficiency units of workers in that submarket. A worker with high ability has a higher probability of forming a productive match than a worker with low ability, but a worker does not precisely know this probability because he does not know his ability. Productive matches face an exogenous probability of separation, in which case the worker reenters the unemployment pool and continues to search.

Success and failure of search both convey useful information about a worker's type. Success in getting a match is good news about a worker's ability, but failure to match generates an effect akin to discouragement. After a worker fails to find a match, he views this outcome as bad news about his ability and, hence, updates his beliefs about ability downward. Subsequently, the worker will choose to search for jobs that will be easier to get. Those jobs will necessarily come with lower wages as part of the equilibrium tradeoff between wages and market tightness. Thus, learning from search induces not only reservation wages, but also desired wages, to decline with unemployment duration. As workers experience different search outcomes, their labor market histories and, hence, their beliefs about their ability diverge. Firms cater to the workers with different beliefs by offering different wages, and so there is a non-degenerate distribution of equilibrium wages among *ex ante* identical workers across submarkets.

Our theory explains the above facts about labor market histories as follows. First, longer unemployment durations result in larger wage losses because unsuccessful search depresses a worker's beliefs about his ability and induces the worker to lower his desired wages in subsequent search. Second, the average job-finding probability decreases with unemployment duration because the average ability in any given cohort of workers decreases with unemployment duration as low-ability workers are relatively less likely to find jobs. Controlling for workers' ability, however, job-finding probabilities increase with unemployment duration. Third, since firms are willing to offer higher wages when they expect to attract more applicants, jobs catering to workers with shorter unemployment durations have higher wages and are filled more rapidly. Fourth, a worker's pre-unemployment wage reflects the worker's beliefs about his ability when entering the current spell of unemployment. Workers with higher pre-unemployment wages are those who had relatively shorter past unemployment spells. Since these workers are relatively more optimistic about their ability, they search for relatively higher wages, with a lower job-finding probability, in the current spell of unemployment, which tends to result in longer unemployment durations.

Learning from search is an intuitive process, but it has not been incorporated into an equilibrium analysis. Instead, the literature has focused on a single agent's optimal stopping problem, resting on the partial equilibrium view of the labor market originated in McCall's (1970) search model. Thus, Burdett and Vishwanath (1988) consider the natural case in which workers learn about an exogenously given distribution of wage offers. However, there are several reasons why an equilibrium framework, in which job search, job creation and wage offers are jointly determined, is important for understanding the consequences of learning from search. First, the aggregation of individuals' behavior may reveal information (e.g., about the wage distribution) that stops individual learning altogether. Second, even if individual learning is consistent with aggregation, firms have an incentive to respond to workers' learning from search by adjusting offers and vacancies. These responses are important for understanding the tensions between aggregate and individual behavior, as reflected for instance by the relationship between wages and the duration of vacancies as well as unemployment. In our analysis, the properties of the equilibrium wage function play an important role for establishing the central result that a worker's desired wages are a strictly increasing function of the worker's beliefs. Third, an equilibrium theory is necessary for understanding how the consequences of unemployment for individuals depend on aggregate economic conditions.

The main difficulty in analyzing learning in an equilibrium stems from the need to address the interaction between individuals' learning and aggregate prices. To appreciate why, it is useful to consider the work of Burdett and Vishwanath (1988), where workers learn about the unknown distribution of wages. In their model, each worker receives a wage offer from an exogenous distribution and then decides whether to accept the offer or to reject it and continue to search. A worker who receives an offer lower than expected revises his beliefs about the wage distribution downward. Workers with longer unemployment spells are precisely the ones who have drawn and rejected relatively lower wages in the past, and so they perceive the jobs available to them as jobs offering low wages. As a result, reservation wages are negatively related to unemployment spells. While this is an interesting result, Burdett and Vishwanath (1988) examine only the workers' side of the labor market. In equilibrium, however, learning by the market participants will affect the wage distribution by affecting firms' decisions about wage offers.

In contrast, our modeling strategy makes it tractable to analyze the relationship between learning, search and wages. First, we propose that incomplete information lies in a worker's intrinsic characteristic – his ability to form a productive match with a job. The advantage of this assumption, relative to learning about the wage distribution, is that this characteristic does not change with learning. Second, we model search as a directed and competitive process, as in Moen (1997) and Acemoglu and Shimer (1999), which allows individuals to sort themselves into submarkets.[2] The advantage of this formulation, relative to random search, is that it makes the equilibrium block recursive in the sense that individuals' decisions and market tightness are independent of the distribution of workers.[3]

In addition to an equilibrium formulation of learning from search, we provide an analytical procedure for resolving a main theoretical problem in the analysis of optimal learning from experience. This problem is caused by convexity of the value function. Because search outcomes generate variations in a worker's posterior beliefs about his ability, search conveys valuable information only if these variations in beliefs are valuable to the worker, that is, if the worker's nonlinear value function is convex in beliefs. Although the literature (e.g., Easley and Kiefer, 1988) recognizes that such convexity is likely to lead to multiple solutions and render the first-order conditions inapplicable, previous work has either ignored the difficulty or focused on corner solutions (e.g. Balvers and Cosimano, 1993). We resolve this difficulty by exploiting a connection between convexity of the value function and the lattice-theoretic property of *supermodularity*.

This connection is not immediately obvious and, to our knowledge, has not been examined. In our model, neither a worker's current payoff nor his objective function is supermodular as is often required in applications (see Topkis, 1998, and Milgrom and Shannon, 1994). We proceed in two steps. First, we use convexity of a worker's value function to show that a particular monotone transformation makes the worker's objective function supermodular. This approach differs considerably from other applications of lattice-theoretic techniques to dynamic programming (e.g., Amir et al., 1991, Mirman et al., 2008), which assume the value function to be concave. Second, we establish the result that workers' optimal decisions (i.e., desired wages) are strictly increasing with the workers' beliefs. Such strict monotonicity is necessary for the model to account for the fact that wage losses increase with unemployment duration. Generally, lattice-theoretic techniques establish only weak monotonicity of optimal decisions. For strict monotonicity, the literature has required strong assumptions on differentiability, e.g., Amir (1996) and Edlin and Shannon (1998). Because such assumptions can be violated here, we establish strict monotonicity in an alternative way, again by exploiting convexity of the value function.

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[2] See Peters (1984, 1991), Burdett et al. (2001) and Shi (2001) for analyses of directed search as a strategic problem that leads to the competitive search equilibrium outcome as the market becomes large.

[3] Shi (2009) first formalizes this notion of block recursive equilibria and proves existence of such equilibria in the context of on-the-job search where firms offer wage-tenure contracts to direct workers' search.

Section 2 presents our model, and section 3 characterizes the equilibrium. Our main results on the monotonicity of desired wages are found in section 4. Section 5 considers the uniqueness of optimal choices, and section 6 determines the stationary distribution of workers. In section 7, we discuss the model's empirical implications and its relationship with the literature. Section 8 concludes. All proofs are relegated to the Appendix.

2. The Model Environment

Time is discrete and all agents discount the future at a rate $r > 0$. The economy is populated by a unit measure of workers. The measure of firms will be determined endogenously by free entry. A worker is either employed or unemployed. When unemployed, a worker searches for a job, and the unemployment benefit per period is $b \geq 0$. When employed in a productive match, a worker produces $y > 0$ units of goods per period. After production, the worker-job pair faces exogenous separation with probability $\delta > 0$.

Each worker has unknown ability that is either high (H) or low (L). Ability is a worker's permanent characteristic, determined at the time when the worker first enters the market. The probability that a new worker has ability $i \in \{H, L\}$ is p_i, where $p_H = p \in (0,1)$ and $p_L = 1 - p$. The ability determines a worker's productivity in the following probabilistic way.[4] With a randomly drawn job, a worker with ability i is productive with probability a_i and not productive with probability $(1 - a_i)$, where $1 > a_H > a_L > 0$. We refer to a_i as a type-i worker's "efficiency" or "productive" units. Since whether a worker is productive or not is observed when the worker contacts a firm, the firm will hire the worker only when the worker is productive at the job.

A worker uses his labor market experience to learn about his ability. If the worker could have an infinitely long history in the labor market, he would eventually be able to learn his true ability. To rule out this uninteresting case, we assume that at the end of each period, a worker (employed or unemployed) is exogenously forced out of the market with probability $\sigma > 0$, in which case the worker's future payoff is normalized to zero. An exiting worker is replaced with a new worker who enters the labor market through unemployment so that the labor force is constant.

The events in a period unfold as follows. First, new workers enter the labor market through unemployment to replace the workers who exogenously exited the market in the

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[4] We are very grateful to Daron Acemoglu and the referees for directing us toward this formulation. In a previous version of the paper (Gonzalez and Shi, 2007), we formulated the problem as one of incomplete information about the characteristics of local labor markets rather than individuals.

previous period. Immediately after a new worker enters the labor market, nature determines the worker's ability, while the ability of a worker who was born in the past remains unchanged. Second, firms and unemployed workers make their search decisions. For simplicity, we assume that workers do not search on the job. Third, after the matching process is completed in the period, matched firms and workers produce. At the end of the period, employed workers face the probability of exogenous job separation, and all workers face the probability of exogenous exit.

Search is competitive, as follows. There is a continuum of *submarkets* indexed by x, which will be linked to matching rates in that submarket. The domain of x is $X = [0, 1/a_H]$. A submarket x is characterized by a wage level, $W(x)$, and a tightness, $\lambda(x)$. The functions $W(.)$ and $\lambda(.)$ are public information, taken as given by agents and determined in the equilibrium. In each period, a worker's or a firm's search decision is to choose x, i.e., the submarket to search.[5] Search is directed in the sense that an agent's choice of a submarket involves a tradeoff between the wage and the tightness, because a submarket with a high wage has relatively fewer vacancies per worker in the equilibrium. Note that in line with the formulations by Moen (1997) and Acemoglu and Shimer (1999), a firm does not directly set wages. Instead, by choosing a submarket, a firm chooses a pair (W, λ) from the menu $\{(W(x), \lambda(x)) : x \in X\}$.[6]

In each submarket, the number of matches is given by a matching function. Since a firm will hire a worker only when the worker is productive at the job, it is useful to specify the matching function to determine the number of productive matches rather than the number of contacts. Let $v(x)$ denote the number of vacancies created in submarket x, and $u_i(x)$ the number of unemployed, type-i workers in submarket x, where $i \in \{H, L\}$. We define the total efficiency/productive units of workers searching in submarket x as

$$u_e(x) = a_H u_H(x) + a_L u_L(x). \tag{2.1}$$

Define the effective tightness in this submarket as $\lambda(x) = v(x)/u_e(x)$. The number of productive matches in submarket x is given by a function, $F(u_e(x), v(x))$. The index x

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[5] Workers who differ in *ex post* beliefs may also choose different levels of search intensity and labor market participation. Although our analysis can shed light on such differences, we choose to abstract from them for the sake of simplicity.

[6] It is inadequate to index the submarkets by the length of the unemployment duration of the participating workers. First, workers who have the same unemployment duration can still be heterogeneous in beliefs about their ability. One source of this heterogeneity is that the workers may have had different employment histories prior to unemployment. Indexing the submarkets by x allows these workers to optimally make different search choices, but indexing the submarkets by the unemployment duration does not. Second, in contrast to the discrete length of the unemployment duration, x is a continuous variable which allows agents to make a continuous tradeoff between W and x.

is the matching rate for each efficiency unit of a worker in submarket x; that is,

$$x = \frac{F\left(u_e\left(x\right), v\left(x\right)\right)}{u_e\left(x\right)}.$$

For a type-i worker in submarket x, the probability of getting a productive match is $a_i x$. Thus, given x, the lower a worker's ability, the lower his matching probability. The matching probability of a vacancy in submarket x is $F/v = x/\lambda(x)$.

We impose the following standard assumption on the matching function F:

Assumption 1. *The function $F(u_e, v)$: (i) is strictly increasing, strictly concave and twice differentiable in each argument, (ii) is linearly homogeneous, and (iii) has $F(u, 0) = 0$ and $F(u, \infty) > 1/a_H$ for all $u \in (0, \infty)$, and $x/\lambda(x) \leq 1$ for all x.*

The tightness $\lambda(x)$ is the solution for λ to the equation $F(1, \lambda) = x$, which satisfies:

$$\lambda'(x) > \frac{\lambda(x)}{x} > 0,\ \lambda''(x) > 0, \text{ for all } x \in (0, 1/a_H]. \tag{2.2}$$

The above properties of λ imply that a vacancy's matching probability, $x/\lambda(x)$, decreases in x. That is, if it is easy for a worker to find a productive match in submarket x, it must be difficult for a firm to find a productive worker in the submarket.

The key feature of the model is the incomplete information about worker ability, which implies that workers who search for jobs face a signal extraction problem. Search histories are informative because low-ability workers are more likely to fail to get matches in any given period. As will be shown below, self-selection of workers into firms according to their own information implies that firms do not need to know the workers' histories.

Let us remark on two implications of the formulation that significantly simplify the analysis of the learning problem in the equilibrium. First, there is no need for any individual (worker or firm) to learn about the composition of high- versus low-ability workers in a submarket. In any submarket x, a worker's matching probability depends only on the worker's own ability and x, while a vacancy's matching probability depends only on x. Thus, given the choice x, an agent's expected payoff in the submarket is independent of the level and the composition of the efficiency units in the submarket. With such independence, free entry of firms into the submarket ensures that the effective tightness and the wage in the submarket are functions only of x and not of the composition of workers. Second, and similarly, firms do not face a signal extraction problem, in contrast to workers. Every aspect of a vacancy's expected payoff is a function of only the choice x.[7]

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[7] These simplifying features depend on the use of directed search, but also on two other aspects of

3. Learning in Competitive Search Equilibrium

3.1. Learning from Search

Workers update beliefs about their ability after each search. The updating depends on the particular submarket into which the worker just searched. To describe the updating process, it is convenient to express a worker's belief in terms of his expectation of a, the probability that he will be productive with a randomly selected job. Let the initial prior expectation of a for a new worker be $\mu_0 \in (a_L, a_H)$. From the distribution of new workers over the levels of a, we can calculate: $\mu_0 = pa_H + (1-p)a_L$, where $p \in (0,1)$. This mean belief is common to all new workers and it is public information.[8]

Consider the updating process for an arbitrary worker. Let P_i be the prior probability with which $a = a_i$, where $a_i \in \{a_H, a_L\}$. Let μ be the expected value of a according to these prior beliefs and refer to μ simply as beliefs. Denote the domain of μ as $M = [a_L, a_H]$. Note that the prior distribution of a is Bernoulli, with mean μ. From the definition of μ, we can solve P_i in terms of μ:

$$P_H = \frac{\mu - a_L}{a_H - a_L}, \qquad P_L = \frac{a_H - \mu}{a_H - a_L}. \tag{3.1}$$

Let $k \in \{0,1\}$ be the matching outcome in the current period, where $k = 0$ if the worker fails to get a match and $k = 1$ if the worker succeeds in getting a match. Then,

$$P(a_i|x,\ k=1) = \frac{a_i}{\mu}P_i, \qquad P(a_i|x,\ k=0) = \frac{1-xa_i}{1-x\mu}P_i. \tag{3.2}$$

The conditional distribution of a is Bernoulli with mean $\mathbb{E}(a|x,k) = a_H P(a_H|x,k) + a_L\left[1 - P(a_H|x,k)\right]$. Substituting $P\left(a_H|x,k\right)$ from (3.2) and P_H from (3.1), we have:

$$\mathbb{E}\left(a|x,k=1\right) = \phi(\mu), \qquad \mathbb{E}(a|x,k=0) = H(x,\mu), \tag{3.3}$$

where

$$\phi(\mu) \equiv a_H + a_L - \frac{a_H a_L}{\mu}, \tag{3.4}$$

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the model. One is that the matching function directly specifies the number of productive matches rather than the number of contacts, and the other aspect is that the numbers of high- and low-ability workers are perfect substitutes for each other in the matching function. If the matching function specifies the number of contacts, instead, or if it has the form $F(u_H(x), u_L(x), v(x))$ where u_H and u_L are not perfect substitutes, then an agent's probability of getting a productive match will depend on the composition, $(u_H(x), u_L(x))$. In either case, the composition of workers in a submarket will affect the expected payoff to a vacancy in the submarket and, hence, the equilibrium wage. By eliminating this complexity, our modeling approach keeps the analysis tractable.

[8] For simplicity we abstract from heterogeneity in the initial beliefs among new workers. However, our model does generate heterogeneous beliefs among workers with different employment histories.

$$H(x,\mu) \equiv a_H - \frac{1 - xa_L}{1 - x\mu}(a_H - \mu). \tag{3.5}$$

Note that if the initial mean belief μ_0 exceeds a_L, $\mathbb{E}(a|x,k) > a_L$ for both $k = 0$ and $k = 1$. Also note that $\phi(\mu) > \mu > H(x,\mu)$ for all $\mu \in (a_L, a_H)$, $\phi'(\mu) > 0$, and $\phi''(\mu) < 0$. Moreover, $H(x,\mu)$ is a decreasing function of x; that is, a higher x reduces the worker's posterior beliefs after the worker fails to find a match.

The updating process above has two preliminary properties. First, the sequence of mean beliefs is a Markov process. Second, a worker's mean belief, μ, is a sufficient statistic for the worker's unemployment history.

The value of x measures the informativeness of search. Intuitively, search outcomes in a market with a relatively higher x are more informative because a market with a higher x has a relatively higher matching probability for a worker; if a worker fails to find a match in such a market, the worker will more likely attribute the failure to low ability. This relationship between the level of x and the informativeness can be precisely captured using the criterion in Blackwell (1951). Consider the information revealed by search in two different submarkets, with $x > x'$. Let K and K' be the random number of matches associated with x and x'. Intuitively, one can construct the random variable K' by "adding noise" to K as follows. First, let the worker randomize with probability of success ax, with $a \in \{a_L, a_H\}$; then, whenever the realization is a success, randomize again with success probability x'/x. The result is a Bernoulli trial with probability of success equal to ax'. In other words, if $x > x'$, the random variable, or experiment, K is sufficient for K' (see DeGroot, 1970, pp433-439).

3.2. A Worker's Value Function

Consider first a worker with beliefs μ who is employed at wage w in a period. Let $J_e(\mu, w)$ denote the worker's value function. After producing and obtaining the wage w, the worker exogenously exits the labor market with probability σ and obtains zero payoff in the future; with probability $(1-\sigma)$, the worker remains in the labor market. With probability δ, the worker exogenously separates from the job into unemployment, in which case the worker's value function from the next period onward will be denoted $V(\mu)$. With probability $(1-\delta)$, the worker remains employed at the wage w in the next period, and his value function from the next period onward will be $J_e(\mu, w)$. Thus, the Bellman equation for J_e is:

$$J_e(\mu, w) = w + (1-\sigma)\left[(1-\delta)\frac{J_e(\mu, w)}{1+r} + \delta\frac{V(\mu)}{1+r}\right].$$

The above equation yields:

$$\frac{J_e(\mu, w)}{1+r} = \frac{1}{A}\left[\frac{w}{1-\sigma} + \delta\frac{V(\mu)}{1+r}\right], \text{ where } A = \frac{r+\sigma}{1-\sigma} + \delta. \tag{3.6}$$

Now consider an unemployed worker who enters a period with belief μ. If he chooses to search in submarket x, the expected probability of finding a (productive) match is $x\mu$. If he fails to find a match, his beliefs are updated downward to $H(x, \mu)$ as defined by (3.5). In this case, the worker will continue to search in the next period, and his value function next period will be $V(H(x, \mu))$. If the worker succeeds in finding a match in the current period, his beliefs are updated upward to $\phi(\mu)$ as defined by (3.4). In this case, the worker can choose whether to accept the match. We will impose Assumption 2 below to guarantee that a worker always accepts a match. Under this assumption, the worker's value function next period will be $J_e(\phi(\mu), W(x))$. We adopt the conventional assumptions that a new match starts to produce in the next period and that a newly matched worker does not experience separation in the same period. Note that such a worker does face the possibility of exogenous exit from the market. Thus, under Assumption 2, the worker's expected payoff of searching in a submarket x, discounted to the current period and excluding the unemployment benefit, is given by $(1-\sigma)R(x, \mu)$, where

$$R(x, \mu) \equiv x\mu\frac{J_e(\phi(\mu), W(x))}{1+r} + (1 - x\mu)\frac{V(H(x, \mu))}{1+r}. \tag{3.7}$$

In this case, V is given by the following Bellman equation:

$$V(\mu) = b + (1-\sigma)\max_{x \in X} R(x, \mu). \tag{3.8}$$

Denote the set of optimal decisions in (3.8) as $G(\mu)$ and a selection from $G(\mu)$ as $g(\mu)$.

When choosing a submarket x, the worker faces two considerations. One is the familiar tradeoff between the wage and the matching probability in models of directed search. That is, a submarket with a higher x has a lower wage and a higher job-finding probability. Another consideration is learning from the search outcome. As discussed earlier, search in a submarket with a high x (i.e., a low wage) is more informative than search in a submarket with a low x. The value of this information is captured by the features of the value function, to be described later in Lemma 3.1 and Theorem 4.2.

It is useful to note that the set of solutions $G(\mu)$ generically contains only a finite number of values. That is, given beliefs μ, a worker prefers to search in only a few submarkets and possibly only one submarket. Over time, the worker switches from one submarket to another not because he is indifferent between these submarkets, but because search outcomes induce the worker to update beliefs.

In principle, workers may have incentive to engage in the following "experimentation": searching during a period solely to gather information and, thus, refusing to enter a match once they learn that a match has occurred. This may occur because a worker who found a match in submarket x will revise his belief upward to $\phi(\mu)$. We do not think that this form of experimentation is important in practice, unless it is associated with heterogeneity among productive matches, which does not exist here. Thus, we rule out such experimentation by focusing on the case in which employment is sufficiently valuable to a worker that the worker always prefers to accept a match that he searches for.

Assumption 2. *Define* x^* *by the solution to* $\lambda'(x^*) = a_H \lambda(\frac{1}{a_H})$ *and note that* $x^* \in (0, 1/a_H)$*. Assume that labor productivity satisfies:*

$$\frac{y-b}{c} > [A + a_H x^*]\, \lambda'(x^*) - a_H \lambda(x^*).$$

This sufficient condition implies that a worker prefers getting the lowest equilibrium wage every period starting now to remaining unemployed in the current period and then getting the highest possible wage from a match starting next period (see Appendix A). Intuitively, the condition requires that the opportunity cost of rejecting a match, as reflected by $(y - b)$, should be sufficiently high to a worker.[9] Stronger than necessary, this condition significantly simplifies the analysis and the exposition of our main results. As in Burdett and Vishwanath (1988), one can relax the condition by introducing a direct cost of search per period, which further increases a worker's opportunity cost of rejecting an offer. For simplicity, however, we have not included such a cost of search.

Remark 1. *Since* $x^* \lambda'(x^*) > \lambda(x^*)$*, Assumption 2 implies:* $y - b > cA\lambda(x^*)/x^*$*, which in turn implies* $y - b > cA\lambda'(0)$*. The last inequality says that there are feasible wages at which employment is better than unemployment for a worker.*

3.3. Free Entry of Firms and the Equilibrium Definition

There is free entry of firms into the market. After incurring a cost $c \in (0, y)$, a firm can post a vacancy for a period in any one of the submarkets. To a firm, the value of a job filled at wage w is

$$J_f(w) = y - w + (1-\sigma)(1-\delta)\frac{J_f(w)}{1+r}. \tag{3.9}$$

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[9] The discount rate in Assumption 2, appearing through A, reflects both workers' and firms' discount rate. For a worker, a higher discount rate lowers the benefit from experimentation for any given wage. However, when firms discount future at a higher rate, the present value of a filled job falls, and wages in all submarkets must be lower in order to induce firms to enter. In this case, the loss of the current wage from experimentation falls. With a common discount rate, the effect through firms' discount rate dominates.

The matching probability for a vacancy in submarket x is $x/\lambda(x)$. The discounted value of the match to the firm is $(1-\sigma)\frac{J_f(W(x))}{1+r}$. Solving J_f from (3.9), we can express a firm's value of a vacancy in submarket x as

$$J_v(x) = -c + \frac{x}{\lambda(x)} \frac{y - W(x)}{A}, \tag{3.10}$$

where A is defined in (3.6). The firm chooses x to maximize $J_v(x)$.

In equilibrium, a firm is willing to enter any submarket, provided that the wage in the submarket is consistent with the free-entry condition. Precisely, the value function, $J_v(x)$, and the number of vacancies, $v(x)$, satisfy: $J_v(x) \leq 0$ and $v(x) \geq 0$ for all $x \in X$, where the two inequalities hold with complementary slackness.[10] Thus, for all x such that $v(x) > 0$, the wage function is:

$$W(x) = y - cA\frac{\lambda(x)}{x}. \tag{3.11}$$

Conversely, for any feasible wage level specified below, we require the number of vacancies to be positive. The wage function has the following properties:

$$\begin{aligned}
&\text{(i) } b + ca_H[x^*\lambda'(x^*) - \lambda(x^*)] \leq W(x) \leq y - cA\lambda'(0), \\
&\text{(ii) } W'(x) < 0, \qquad \text{(iii) } 2W'(x) + xW''(x) < 0.
\end{aligned} \tag{3.12}$$

Part (i) specifies the interval of feasible wages, where x^* is defined in Assumption 2. The upper bound on wages comes from the fact that $\lambda(x)/x \geq \lambda'(0)$. The lower bound on wages comes from Assumption 2 and the fact that $\lambda(x)/x \leq a_H\lambda(\frac{1}{a_H}) = \lambda'(x^*)$. The lower bound on wages is strictly greater than b because $x^*\lambda'(x^*) > \lambda(x^*)$. Also, Assumption 2 is sufficient for the wage interval in (i) to be non-empty.

Parts (ii) and (iii) of (3.12) are implied by (2.2), which is in turn implied by Assumption 1 on the matching function. Part (ii) says that a higher employment probability occurs together with a lower wage. This negative relationship is necessary for providing a meaningful tradeoff between the two variables in directed search. As such, part (ii) is necessary for inducing firms to enter the submarket. Part (iii) is implied by $\lambda''(x) > 0$, and it says that the function $xW(x)$ is strictly concave in x.

Focus on stationary symmetric equilibria. Such an *equilibrium* consists of workers' choices of x, a wage function, $W(x)$, value functions, (J_e, V, J_f, J_v), and a sequence of

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[10] Although there is a first-order condition for the maximization problem $\max_x J_v(x)$, the condition is a differential equation for the wage function. Without an initial condition, this equation has a continuum of solutions, which means that there is a continuum of choices of x that are optimal for the firm.

beliefs that meet the following requirements. (i) Given the wage function, all workers with the same belief μ at the beginning of a period use the same optimal search policy $x = g(\mu) \in G(\mu)$ that solves (3.8). (ii) A worker with beliefs μ updates beliefs according to $\phi(\mu)$ upon getting a match and according to $H(g(\mu), \mu)$ upon failing to get a match. (iii) The value functions satisfy (3.6), (3.8), (3.9) and (3.10). (iv) Free-entry: the wage function $W(x)$ satisfies (3.11) for all x such that $W(x)$ satisfies (i) of (3.12). (v) Consistency: for every submarket x with positive entry, the mass of all vacancies in x divided by the efficiency units of workers who choose x is equal to $\lambda(x)$.

In the above definition, we have left out the steady-state conditions on worker flows and the wage distribution, which will be characterized in section 6. We deliberately do so in order to emphasize "block recursivity" of an equilibrium in our model, which stands for the property that individuals' decisions and matching probabilities are independent of the distributions of workers and wages. To analyze these decisions and matching probabilities, all that is required is the wage function $W(.)$ and the tightness function $\lambda(.)$, which are determined by firms' free-entry condition and the matching function. After completing this analysis, we can simply aggregate individuals' decisions to find the equilibrium distributions of workers and wages. Block recursivity makes the analysis tractable by reducing the dimensionality of the state variables for individuals' decision problems significantly. As first formulated and discussed by Shi (2009), block recursivity is a feature of directed search which allows workers to sort into submarkets. In our model, the workers sort according to their beliefs about their ability. Since each submarket attract only the workers with particular beliefs, firms that post vacancies in that submarket calculate the expected profit with only such workers in mind – they do not need to consider how other workers with different beliefs are distributed. Free entry of firms will guarantee that each submarket will have exactly the effective tightness specified for that submarket. If search were undirected, instead, an individual's search decision would depend on the wage distribution which, in turn, would evolve as individuals learn about their ability.

3.4. Existence of an Equilibrium

Let us analyze a worker's problem, (3.8). It is easy to see that the right-hand side of (3.8) is a contraction mapping on V. Using (3.12), standard arguments show that a unique value function V exists, which is positive, bounded and continuous on $M = [a_L, a_H]$ (see Stokey and Lucas with Prescott, 1989, p79). Moreover, the set of maximizers, G, is nonempty, closed, and upper hemi-continuous. The following lemma summarizes the existence result and some other features of the equilibrium (see Appendix A for a proof):

Lemma 3.1. *Under Assumptions 1 and 2, there exists an equilibrium where all matches are accepted. In the equilibrium,* $g(\mu) > 0$ *for all* $g(\mu) \in G(\mu)$ *and all* $\mu \in M$*. Moreover,* V *is strictly increasing, (weakly) convex, and almost everywhere differentiable.*

Let us explain the features of g and V in the above lemma. First, optimal choices of x are strictly positive. A worker who chooses $x = 0$ has zero probability of finding a match and does not learn anything from the search (i.e., $H(0, \mu) = \mu$). Since there are feasible wages at which employment is strictly better than unemployment (see Remark 1), a worker will choose $x > 0$. Second, the value function of an unemployed worker is strictly increasing in the worker's beliefs. A worker with higher beliefs can always choose to enter the same submarket as does a worker with lower beliefs and, thereby, can obtain a match with a higher expected probability. This translates into a higher expected payoff for the worker. Third, the value function is (weakly) convex in the worker's beliefs, as is standard in the literature of optimal learning (see Nyarko, 1994). Search generates information by creating variations in the worker's posterior beliefs. Such variations can never be harmful to the worker because the worker can always choose to ignore the information. Weak convexity of the value function reflects this fact.

A worker's *reservation wage* can be defined in the conventional way as the lowest permanent income that a worker will accept to forego search. This is given as $\frac{r+\sigma}{1+r}V(\mu)$.[11] Monotonicity of the value function determines the behavior of reservation wages. Because $V(\mu)$ is strictly increasing, the reservation wage strictly falls over each unemployment spell as the worker's beliefs about his own ability deteriorate. Put differently, a worker's permanent income strictly declines over each unemployment spell. Similarly, with strict monotonicity of V, (3.8) implies that a worker's reservation wage is always strictly lower than the desired wage, i.e., $\frac{r+\sigma}{1+r}V(\mu) < W(g(\mu))$ for all $\mu > a_L$.

Our focus is on a worker's *desired wage*, which is defined as $w(\mu) = W(g(\mu))$. In contrast to reservation wages, desired wages are much more difficult to analyze because they depend on optimal learning from search. As it will become clear in the next section, monotonicity of the optimal search decision relies crucially on convexity of the value function.

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[11] For a worker who gets income w in every period, starting in the current period, and who is subject to exogenous exit and time discounting, the present value is $V = w + \frac{1-\sigma}{1+r}V$, which yields $V = \frac{1+r}{r+\sigma}w$. Thus, the permanent income corresponding to a present value V is $\frac{r+\sigma}{1+r}V$.

4. Monotonicity of Workers' Desired Wages

In this section, we establish the result that the workers' desired wage, $w(\mu)$, is an increasing function of beliefs. Because $w(\mu) = W(g(\mu))$, where $W(.)$ is decreasing, monotonicity of $w(\mu)$ is equivalent to the feature that the worker's optimal choice of the submarket, $x = g(\mu)$, is decreasing in μ. Let us define $z = -x$ and refer to z, rather than x, as the worker's search decision. This transformation will be useful for what follows, and it enables us to attach the label *monotone decisions* naturally to the feature that z increases in the beliefs. After the transformation, the objective function in (3.8) becomes $R(-z, \mu)$, and the feasible set of choices is $-X \ni z$. The domain of μ is $M = [a_L, a_H]$.

Before undertaking the task, note that convexity of the value function in beliefs, which is inherent to learning, implies that one cannot guarantee in general that the objective function $R(-z, \mu)$ is single-valued or differentiable. Thus, the optimal decision is not necessarily unique or interior, and the first-order condition may not be applicable.[12] Although these difficulties are well known in the literature on optimal learning (e.g., Easley and Kiefer, 1988), this literature has either ignored them or focused on corner solutions (e.g., Balvers and Cosimano, 1993). We need to examine all solutions in order to characterize optimal search behavior and desired wages.

In the presence of these difficulties, a natural way to establish monotonicity of the workers' search decision is to use lattice-theoretic techniques (see Topkis, 1998). In our model, supermodularity of the objective function is equivalent to the feature of increasing differences in (z, μ), because these variables lie in closed intervals of the real line.[13] However, the connection between supermodularity and the dynamic programming problem in (3.8) is far from obvious. In (3.8), the value function is convex and the immediately payoff, $-\mu z W(-z)$, is not supermodular in (μ, z). The opposite features are imposed in other applications of supermodularity to dynamic programming, which use concavity of the value function and supermodularity of the current payoff function, recursively via the Bellman equation, to establish that the objective function is supermodular.[14]

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[12] In different modeling environments, there are techniques to generate smooth optimal choices and differentiable value functions, e.g., Santos (1991). However, those techniques require the value function to be concave, which is violated here.

[13] Let $z \in Z$ and $\mu \in M$, where Z and M are partially ordered sets. A function $f(z, \mu)$ has increasing differences in (z, μ) if $f(z_1, \mu_1) - f(z_1, \mu_2) \geq f(z_2, \mu_1) - f(z_2, \mu_2)$ for all $z_1 > z_2$ and $\mu_1 > \mu_2$. If the inequality is strict, then f has strictly increasing differences. In our model, Z, M and $Z \times M$ (under the product order) are all lattices. In this case, the feature of increasing differences implies supermodularity (see Topkis, 1998, p.45).

[14] Examples include Amir et al. (1991), Mirman et al. (2008), and Becker and Boyd (1997, pp. 277-284).

However, monotonicity of optimal choices is invariant to transformations of the objective function that are monotone in the choice variables. Although the objective function $R(-z,\mu)$ is unlikely to be supermodular in (μ, z), it is possible to transform the objective function into a supermodular function. To that end, we express (3.8) as

$$V(\mu) = b + (1-\sigma)\,\mu \max_{z \in -X} \hat{R}(z,\mu),$$

where $\hat{R}$ is defined as $\hat{R}(z,\mu) \equiv \frac{1}{\mu} R(-z,\mu)$, that is,

$$\hat{R}(z,\mu) = -\frac{z}{A}\left[\frac{W(-z)}{1-\sigma} + \delta \frac{V(\phi(\mu))}{1+r}\right] + \left(z + \frac{1}{\mu}\right)\frac{V(H(-z,\mu))}{1+r}. \tag{4.1}$$

Denote $Z(\mu) = \arg\max_{z \in -X} \hat{R}(z,\mu)$ and $z(\mu) \in Z(\mu)$. Clearly, the set of optimal choices for x is $G(\mu) = -Z(\mu)$, and a typical selection is $g(\mu) = -z(\mu)$. Denote the greatest selection of $Z(\mu)$ as $\bar{z}(\mu)$ and the least selection as $\underline{z}(\mu)$.

We state the following assumption and theorem (see Appendix C for a proof):

Assumption 3. *The job separation rate satisfies* $0 < \delta \leq \bar{\delta}$, *where* $\bar{\delta} > 0$ *is defined in part (5) of Appendix B.*

Theorem 4.1. *Let* $z \in -X$ *and* $\mu \in M$. *Under Assumptions 1, 2 and 3, the function* $\hat{R}(z,\mu)$ *is strictly supermodular in* (z,μ). *In this case, every selection* $z(\mu)$ *is an increasing function; that is, for all* μ_a *and* μ_b *in* M, *with* $\mu_a > \mu_b$, *it is true that* $z(\mu_a) \geq z(\mu_b)$ *for all* $z(\mu_a) \in Z(\mu_a)$ *and all* $z(\mu_b) \in Z(\mu_b)$. *Similarly, every selection* $g(\mu)$ *from* $G(\mu)$ *is a decreasing function, and the wage* $w(\mu)$ *is an increasing function.*

The main task in the proof of this theorem is to establish strict supermodularity of $\hat{R}$, after which monotonicity of $z(\mu)$ follows from Topkis (1998, p79) and Milgrom and Shannon (1994).[15] Two aspects of the proof are worth noting. First, every solution for z is an increasing function of the beliefs. This strong result comes from the feature that $\hat{R}(z,\mu)$ is strictly supermodular. Second, convexity of the value function plays an important role for strict supermodularity of $\hat{R}$ and, hence, for monotone optimal choices.

To understand the roles of Assumption 3 and convexity of V, recall that a worker updates his beliefs to $H(-z,\mu)$ after a match failure and to $\phi(\mu)$ after a match success. The

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[15] It is straightforward to verify that if $\hat{R}(z,\mu)$ is strictly supermodular, then the original function $R(-z,\mu)$ has the strict single-crossing property in (z,μ) as defined by Milgrom and Shannon (1994). Thus, Theorem 4.1 implies that Theorem 4$'$ in Milgrom and Shannon (1994) holds in our model.

expectation of these posterior beliefs is equal to the prior beliefs, μ, and this relationship between the posterior and prior beliefs can be rewritten as

$$(z + \frac{1}{\mu})H(-z, \mu) - z\phi(\mu) = 1.$$

With the expressions for ϕ and H in (3.4) and (3.5), it is easy to see that the first term above is strictly supermodular in (z, μ) and the second term $(-z\phi(\mu))$ is strictly submodular in (z, μ). A convex value function preserves these features. That is, in the transformed payoff $\hat{R}$, the term associated with a match failure, $(z + \frac{1}{\mu})V(H(-z, \mu))$, is strictly supermodular in (z, μ), and the term associated with a match success, $-zV(\phi(\mu))$, is strictly submodular in (z, μ). Because of these opposite features, the payoff function is not supermodular in general. However, it is supermodular under Assumption 3. To see why, note that the two opposing terms described above are not valued equally in a worker's payoff. While the worker will fully use the information contained in a match failure in future search, he will use the information contained in a match success only when he will become unemployed again in the future. The frequency with which the latter event occurs is δ. By putting an upper bound on δ, Assumption 3 limits the use of the information contained in a match success and, hence, limits the effect of the submodularity of the term, $-zV(\phi(\mu))$. Under this assumption, the transformed payoff function is strictly supermodular.

Weak monotonicity of optimal choices is a general result that holds even when optimal choices are corner solutions and when there are multiple solutions. However, weak monotonicity alone cannot capture the phenomenon described in the Introduction that an unemployed worker's wage loss upon reemployment strictly increases with unemployment duration. To capture this phenomenon, a worker's desired wages must strictly increase in beliefs which, in turn, requires a worker's optimal choice of z to strictly increase in beliefs. In general, strict monotonicity of optimal choices requires strong conditions.[16] In contrast, only a very mild condition is needed in our model. The following theorem specifies this condition and establishes that strict monotonicity of optimal choices is equivalent to strict convexity of the value function (see Appendix D for a proof):

Theorem 4.2. *Maintain Assumptions 1, 2 and 3. The following statements are all equivalent to each other: (i) $V(\mu)$ is strictly convex for all μ; (ii) Every selection $z(\mu) \in Z(\mu)$ is*

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[16] Amir (1996) and Edlin and Shannon (1998) also establish strict monotonicity of optimal choices but, in our model, their methods would require the value function to be continuously differentiable. In particular, Edlin and Shannon (1998) assume that the objective function, $\hat{R}(z, \mu)$, has increasing marginal differences. To compute marginal differences, $\hat{R}(z, \mu)$ must be continuously differentiable with respect to z. Because $\hat{R}$ depends on z through the future value function, as well as W, it is differentiable with respect to z only if the value function is so.

strictly increasing in μ*; (iii) For all* $\mu > a_L$, $\{-1/a_H\} \notin Z(\mu)$*; (iv)* $\{-1/a_H\} \notin Z(a_H)$*; (v) The following condition holds:*

$$\frac{y-b}{c} < (A+1)\lambda'(\frac{1}{a_H}) - a_H\lambda(\frac{1}{a_H}). \tag{4.2}$$

Desired wages are strictly increasing functions of beliefs and every selection $g(\mu)$ from $G(\mu)$ is a strictly decreasing function if and only if condition (4.2) holds. Equivalent to statement (iv), this condition requires that a worker with the most optimistic beliefs $\mu = a_H$ should find it not optimal to search in the submarket with the lowest wage, i.e., the submarket with the highest matching probability and the lowest z. If (4.2) is not satisfied, then it is optimal for a worker to search for the lowest wage, regardless of his beliefs. In this sense, (4.2) can be viewed as a regularity condition for learning to be a useful explanation for the fact that wage losses upon reemployment increase with unemployment duration.[17]

To see the role of this regularity condition, let us first explain the equivalence between statements (i) and (ii). This equivalence relies on the following standard property of optimal learning: The value function V is strictly convex in beliefs if and only if there do not exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice x_0 such that x_0 is optimal for all $\mu \in [\mu_b, \mu_a]$ (see Nyarko, 1994). Because every selection $z(\mu)$ is weakly increasing, as established earlier, this standard property implies that the value function is strictly convex if and only if every selection $z(\mu)$ is strictly increasing.

It is easy to see that statement (ii) implies (iii) which, in turn, implies (iv). Thus, the key step in the proof of the above theorem is to show that statement (iv) implies (i). That is, if the value function is not strictly convex, then it is optimal for a worker with the most optimistic beliefs to search in the submarket with the lowest wage. To explain this result, suppose that the value function is not strictly convex. In this case, there is a subinterval $[\mu_b, \mu_a]$, with $\mu_a > \mu_b$, such that the optimal choice is the same for all μ in this subinterval. This must mean that when beliefs lie in the interior of this subinterval, local variations in the positive or negative signal are not valuable to the worker. In particular, the value function must be linear in the subinterval $[\phi(\mu_b), \phi(\mu_a)]$, in which posterior beliefs will lie when the worker finds a match. With such linearity of the value function over future beliefs, strict concavity of the function $[-zW(-z)]$ implies that the payoff function is strictly concave in z and, hence, the optimal choice of z is unique for all $\mu \in (\mu_b, \mu_a)$. For this unique choice to be invariant in μ for $\mu \in (\mu_b, \mu_a)$, it must be at the corner

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[17] The condition in the theorem can hold simultaneously with Assumption 2. To see this, note that the right-hand side of the condition in Assumption 2 is strictly increasing in x^* and, hence, is less than the right-hand side of the condition given in the theorem (since $x^* < 1/a_H$).

$z = -1/a_H$; otherwise, strict supermodularity of the transformed payoff function, $\hat{R}(z, \mu)$, in (z, μ) would imply that the optimal choice should strictly increase in μ. Repeating the above argument, we know that for any positive integer i, the value function must be linear in the subinterval $[\phi^i(\mu_b), \phi^i(\mu_a)]$ and that the optimal choice under such beliefs must be the singleton $\{-1/a_H\}$, where ϕ^i is defined as $\phi^i(.) = \phi(\phi^{i-1}(.))$. Because $\phi^i(\mu)$ converges to a_H for all $\mu \in (a_L, a_H)$, the choice $\{-1/a_H\}$ must also be optimal when $\mu = a_H$.

The above explanation for why (iv) implies (i) has implicitly used the assumption $\delta > 0$.[18] If $\delta = 0$, instead, the information revealed by a match success is not valuable to the worker, because the worker will never be unemployed again. In this case, the above induction does not apply. For some beliefs $\mu = \mu_a > a_L$, the worker may find it optimal to choose $z = -1/a_H$. Once this happens, it is optimal for the worker to choose $z = -1/a_H$ for all beliefs $\mu \leq \mu_a$, in which case the value function is linear in the subinterval $[a_L, \mu_a]$.

It is useful to clarify the role of the equilibrium wage function for the results obtained so far. In contrast to a model of a partial equilibrium or decision theory, our model requires that the wage in each submarket be consistent with free entry of firms. This equilibrium requirement results in the wage function $W(-z)$, as given by (3.11), which links the features of wages to the matching technology. Given the standard assumptions on the matching technology in Assumption 1, the wage function has the properties listed in (3.12). These properties of the wage function are not important for strict supermodularity of the function $\hat{R}(z, \mu)$, which requires only that the value function V be weakly convex. As a result, the equilibrium wage function is not important for the optimal search choice $z(\mu)$ and desired wages $w(\mu)$ to be weakly increasing in beliefs. However, the equilibrium wage function is critical for the optimal search choice and desired wages to be *strictly* monotone in beliefs. In particular, in the above explanation for why statement (iv) implies statement (i), we have explicitly used the property (iii) in (3.12) that the function $[-zW'(-z)]$ is strictly concave. If the wage function were exogenous, or if it had no connection to the matching technology, it would not be clear how it should satisfy (3.12). In this sense, the equilibrium structure of the model is essential for our analysis to capture the intuitive link between learning from search and discouragement.

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[18] The equivalence between statements (i) and (ii) in Theorem 4.2 does not require $\delta > 0$. However, the equivalence between these two statements and other statements does require $\delta > 0$.

5. Uniqueness of the Optimal Paths

Theorem 4.2 above provides (4.2) as the necessary and sufficient condition for every selection to be interior for all beliefs $\mu > a_L$. In this section, we take advantage of this feature to explore further properties of the optimal choices. In particular, we show that even if optimal choices are not unique there is some discipline on the set of paths of optimal choices. Since we focus on symmetric equilibria, all searching workers with beliefs μ use the same selection $z(\mu)$. In Appendix E, we establish the following lemma.

Lemma 5.1. *Maintain Assumptions 1 – 3 and assume (4.2). Denote* $h(\mu) = H(-z(\mu), \mu)$. *An unemployed worker's optimal choices are interior for all* $\mu > a_L$. *Moreover, for all* $\mu \in (a_L, a_H)$, *the derivative* $V'(h(\mu))$ *exists for all* $z(\mu) \in Z(\mu)$, *in which case optimal choices obey the first-order condition,* $\hat{R}_1(z(\mu), \mu) = 0$, *where*

$$\begin{aligned}\hat{R}_1(z(\mu),\mu) = \;& \tfrac{1}{A}\left[\tfrac{z(\mu)W'(-z(\mu)) - W(-z(\mu))}{1-\sigma} - \delta\tfrac{V(\phi(\mu))}{1+r}\right] \\ & + \tfrac{V(h(\mu))}{1+r} - \left(\tfrac{1}{\mu} + z(\mu)\right)\tfrac{V'(h(\mu))}{1+r} H_1(-z(\mu),\mu).\end{aligned}$$

This lemma describes a limited sense of differentiability of the value function: With any interior prior beliefs, the value function is differentiable at posterior beliefs that are induced by optimal choices in the case of a match failure. We will explain this result below. The value function may still fail to be differentiable in the first period, or after a history that contains no failure in finding a match, or off the optimal paths. Despite this fact, the limited sense of differentiability is enough for the first-order condition to be applicable in every period.

Let us now turn to the set of optimal paths. For any subset $S \subseteq M$, denote $\phi(S) = \{\phi(\mu) : \mu \in S\}$ and $h(S) = \{h(\mu) : \mu \in S\}$. For any $\mu \in M$, let us generate a sequence of sets, $\Upsilon(\mu) = \{\Upsilon_i(\mu)\}_{i=0}^{\infty}$, by $\Upsilon_0(\mu) = \{\mu\}$ and $\Upsilon_{i+1}(\mu) = \{\phi(\Upsilon_i(\mu)), h(\Upsilon_i(\mu))\}$ for $i = 0, 1, ..., \infty$. We call $\Upsilon(\mu)$ the tree of equilibrium beliefs generated from μ and $\Upsilon_i(\mu)$ the ith layer of the tree. Given μ' and $z(\mu')$ in the current period, beliefs in the next period will reach the node $\phi(\mu')$ with mean probability $-z(\mu')\mu'$ and the node $h(\mu')$ with mean probability $[1 + z(\mu')\mu']$. Because there are fewer branches following a node when the optimal choice is unique than if multiple choices are optimal, the equilibrium is more predictable in the former case. To analyze the set of equilibrium path, we establish the following link between multiplicity of optimal choices and differentiability of the value function at all possible beliefs (see Appendix F for a proof):

Lemma 5.2. *Maintain Assumptions 1 – 3 and assume (4.2). For any μ_a in the interior of (a_L, a_H), let μ_a^+ denote the limit to μ_a from the right and μ_a^- the limit from the left. Then, $V'(\mu_a^+) = (1-\sigma)R_2(-\bar{z}(\mu_a), \mu_a^+)$ and $V'(\mu_a^-) = (1-\sigma)R_2(-\underline{z}(\mu_a), \mu_a^-)$. Moreover, V is differentiable at μ_a if and only if V is differentiable at $\phi(\mu_a)$ and $\bar{z}(\mu_a) = \underline{z}(\mu_a)$.*

The above lemma says that at arbitrary beliefs $\mu \in (a_L, a_H)$, the value function is differentiable if and only if the optimal choice is unique and the value function is differentiable at posterior beliefs reached by a successful match. The following theorem characterizes the entire set of paths of beliefs and optimal choices (see Appendix F for a proof):

Theorem 5.3. *Maintain Assumptions 1 – 3 and assume (4.2). Consider arbitrary initial beliefs $\mu_0 \in (a_L, a_H)$ and the tree of equilibrium beliefs generated from μ_0, $\Upsilon(\mu_0)$. For any $\mu \in \Upsilon(\mu_0)$ at which V is differentiable, the optimal choice $z(\mu')$ is unique and the value function $V(\mu')$ is differentiable at all $\mu' \in \Upsilon(\mu)$. In particular, the optimal choice is unique and the value function is differentiable at all nodes on the tree $\Upsilon(h(\mu))$ for all $\mu \in \Upsilon(\mu_0)$.*

The above theorem implies that multiple choices are optimal only in the case where the worker's initial prior beliefs lie in the (measure-zero) set in which the value function is not differentiable and where the worker has never had a match failure. If a worker's beliefs ever reach a node at which the value function is differentiable, the optimal choice will be unique and the value function will be differentiable in all periods starting from that node. Lemma 5.1 provides a special case in which a worker's beliefs will reach a node where the value function is differentiable – a node reached by a match failure.

To understand why a match failure induces a tree of beliefs at which the value function is differentiable, consider an arbitrary period where the worker's prior beliefs are μ and let the beliefs following a match failure be $\mu' = h(\mu) = H(-z(\mu), \mu)$. Suppose counterfactually that the worker's choice is such that the value function is not differentiable at μ'. In this case, multiple choices are optimal under the beliefs μ, which induce the left derivative of $V(\mu)$ to be lower than its right derivative. The derivative of the future value function captures an opportunity cost of learning bad news. Thus, the discrete fall in $V'(\mu)$ from the right side of μ to the left side implies that learning slightly more about one's ability in the current period increases the cost of learning by a discrete amount. The worker can avoid this discretely larger cost by choosing z slightly above $z(\mu)$, which will keep the posterior slightly above μ'. In contrast to this discrete increase in the benefit, the increase in the cost of z is a marginal reduction in the matching probability. Thus, the net gain from increasing z slightly above $z(\mu)$ is positive. This contradicts the optimality of $z(\mu)$.

The above explanation also clarifies why multiple choices may remain optimal in period t if a worker started with an initial prior μ_0 at which the value function is not differentiable and if the worker has never failed to find a match up to t. In this case, the path of realized beliefs is $\{\phi^{\tau}(\mu_0)\}_{\tau=0}^{t}$. Because a worker's posterior following a match success, $\phi(\mu)$, is independent of the choice $z(\mu)$, the worker could not have averted the non-differentiability of the value function along this path by modifying the search choice.

6. Steady State Distributions and Worker Flows

We now determine the distribution of workers in the stationary equilibrium. Recall that $\Upsilon(\mu)$ is the tree of equilibrium beliefs generated from μ. Since all newborn workers have beliefs μ_0, and since there is exogenous exit from the market, the support of the stationary distribution of workers is $\Upsilon(\mu_0)$. Immediately before the labor market opens in a period, let $e_i(\mu)$ be the measure of employed workers with beliefs μ and type $i \in \{L, H\}$, and let $\hat{u}_i(\mu)$ be the measure of unemployed workers with beliefs μ and type i. The stationary distribution of workers over beliefs is $\{(e_H(\mu), e_L(\mu), \hat{u}_H(\mu), \hat{u}_L(\mu)) : \mu \in \Upsilon(\mu_0)\}$.

Consider unemployed workers of type $i \in \{H, L\}$. There are three cases. One is that the unemployed workers are newborns (with beliefs μ_0). The measure of newborns with type i is:

$$\hat{u}_i(\mu_0) = \sigma p_i, \tag{6.1}$$

where $p_H = p$ and $p_L = 1 - p$. The outflow from and inflow into this group are both equal to σp_i, and so stationarity always holds for this group.

The second case of unemployed workers of type i is that these workers were unemployed in the previous period, in which case their beliefs in the current period are $h(\mu) = H(-z(\mu), \mu)$ for some $\mu \in \Upsilon(\mu_0)$. All of these workers will move out of the group in the period. The inflow will come from type i unemployed workers with beliefs μ in the current period. A fraction $[1 - a_i g(\mu)]$ of these workers fail to find a match in the current period, in which case they update beliefs to $h(\mu)$. Of these unmatched workers, a fraction $(1 - \sigma)$ will survive the exogenous exit. Thus, stationarity requires:

$$\hat{u}_i(h(\mu)) = (1 - \sigma)[1 - a_i g(\mu)]\hat{u}_i(\mu), \quad \mu \in \Upsilon(\mu_0). \tag{6.2}$$

The third case of unemployed workers of type i is that these workers separated from their jobs in the previous period, in which case their beliefs in the current period are $\phi(\mu)$

for some $\mu \in \Upsilon(\mu_0)$. Again, all of these workers will move out of the group in the period. The inflow will come from job separation of type i employed workers with beliefs $\phi(\mu)$. Thus, stationarity requires:

$$\hat{u}_i(\phi(\mu)) = (1-\sigma)\delta e_i(\phi(\mu)), \quad \mu \in \Upsilon(\mu_0). \tag{6.3}$$

Similarly, consider employed workers of type $i \in \{H, L\}$ with beliefs $\phi(\mu)$, where $\mu \in \Upsilon(\mu_0)$. A fraction σ of these workers will exit the market exogenously. Among those who survive the exit, a fraction δ will separate from the job and become unemployed. Thus, the outflow from the group in the period is $[\sigma + (1-\sigma)\delta]\, e_i(\phi(\mu))$. The inflow will come from unemployed workers of type i with beliefs μ. A fraction $a_i g(\mu)$ of these unemployed workers find a match, in which case they update beliefs to $\phi(\mu)$. Of these matched workers, a fraction $(1-\sigma)$ will survive the exogenous exit. Thus, stationarity requires:

$$[\sigma + (1-\sigma)\delta]\, e_i(\phi(\mu)) = (1-\sigma)a_i g(\mu)\hat{u}_i(\mu), \quad \mu \in \Upsilon(\mu_0). \tag{6.4}$$

The equations (6.1) – (6.4) determine the stationary distribution. This task is straightforward because the equilibrium is block recursive. That is, optimal choices are independent of the distribution, as characterized in previous sections. In fact, we can solve for the distribution by going through the nodes of the tree. At the root of the tree, μ_0, (6.1) solves for $\hat{u}_H(\mu_0)$ and $\hat{u}(\mu_0)$. On the first layer of the tree, the beliefs are $(\phi(\mu_0), h(\mu_0))$. Substituting the solutions obtained at μ_0 and setting $\mu = \mu_0$ in (6.2) – (6.4), we can solve for the distribution of workers, $\{\hat{u}_i(h(\mu_0)), \hat{u}_i(\phi(\mu_0)), e_i(\phi(\mu_0))\}_{i=H,L}$. On the second layer of the tree, the nodes of beliefs are $(\phi^2(\mu_0), h(\phi(\mu_0)), \phi(h(\mu_0)), h^2(\mu_0))$. Workers on this layer are in the groups with the following measures: $e_i(\phi^2(\mu_0))$, $\hat{u}_i(\phi^2(\mu_0))$, $\hat{u}_i(h(\phi(\mu_0)))$, $e_i(\phi(h(\mu_0))$, $\hat{u}_i(\phi(h(\mu_0)))$, and $\hat{u}_i(h^2(\mu_0))$, where $i = H, L$. We can solve for these measures by setting $\mu = \phi(\mu_0)$ and $\mu = h(\mu_0)$ in (6.2) – (6.4). Repeating this procedure, we can solve for the distribution of workers on all layers of the tree. Given the equilibrium tree of beliefs, $\Upsilon(\mu_0)$, the stationary distribution of workers over such beliefs is unique.

In the stationary equilibrium, the set of active submarkets is $\{g(\mu) : \mu \in \Upsilon(\mu_0)\}$. In submarket $g(\mu)$, the measure of type-H workers is $u_H(g(\mu)) = \hat{u}_H(\mu)$ and the measure of type-L workers is $u_L(g(\mu)) = \hat{u}_L(\mu)$. The total number of matches in this submarket is $[a_H\hat{u}_H(\mu) + a_L\hat{u}_L(\mu)]g(\mu)$. The average job-finding probability in submarket $g(\mu)$ is:

$$f(g(\mu)) = \frac{a_H\hat{u}_H(\mu) + a_L\hat{u}_L(\mu)}{\hat{u}_H(\mu) + \hat{u}_L(\mu)} g(\mu). \tag{6.5}$$

Given μ, this probability is constant over time because the composition of workers in the submarket is constant in the stationary equilibrium. Similarly, the average job-finding probability in the entire economy is constant over time.

7. Empirical Implications and Related Literature

Although the main contribution of this paper is theoretical, our analysis can help understand a broad set of empirical facts about the labor market. Before listing these facts below, note that conventional explanations for the negative consequences of unemployment focus on skill differences across workers (Lockwood, 1991) or the loss of skill during unemployment (Pissarides, 1992). It is unlikely that the loss of skill alone is the source of the sharp fall in job-finding probabilities over the first few months of an unemployment spell.[19] Learning from search can help understand why workers' labor market histories have large and persistent effects on their future labor market outcomes, even after controlling for worker characteristics that are supposed to account for variations in skills.

(i) *Search histories and endogenous heterogeneity.* Our theory shows that when workers have imprecise information about their ability in the labor market, search histories endogenously generate heterogeneity in workers' beliefs about their ability. Such endogenous heterogeneity provides a rational explanation for discouragement as the consequence of negative search outcomes. Optimal learning from search implies that a worker's beliefs about his own ability deteriorate with unemployment duration. Such discouragement is reflected in wage losses at reemployment, providing a natural explanation for the negative effect of unemployment duration on future wages that is found in empirical analyses (e.g., Addison and Portugal, 1989, Gregg and Wadsworth, 2000, and Nickell et al., 2002).

A further implication of optimal learning from search is that an unemployed worker's search decision depends on the worker's beliefs about his ability which, in turn, are determined by the worker's entire history of search. This implication naturally suggests that an empirical investigation of job-finding probabilities and reemployment wages should take into account not only the worker's most recent unemployment spell, as it is typically done in the empirical literature, but also the worker's previous unemployment spells. Our theory suggests a simple empirical strategy to take into account worker's labor market history. Because a worker's mean beliefs follow a Markov process, the effect of past labor market history is summarized by the worker's beliefs when entering the most recent unemployment spell. In turn, the latter beliefs have a monotone relationship to the worker's wage at the most recent job. Thus, a worker's pre-unemployment wage summarizes the worker's previous experience in the labor market.

The above result provides an explanation for Addison and Portugal's (1989) finding

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[19] Another mechanism that affects unemployed workers' search behavior is the dependence of unemployment benefits on unemployment duration. Burdett (1977) and Mortensen (1977) consider this mechanism.

that unemployment duration increases with pre-unemployment wages after controlling for skills. Workers with higher pre-unemployment wages are those who had relatively shorter durations in previous unemployment spells and, hence, are more optimistic about their ability when entering the current unemployment spell. Controlling for ability, these workers will search for jobs offering higher wages, which are relatively harder to get.

(ii) *Duration dependence.* It is well known that the average job-finding probability falls with unemployment duration (e.g., Kaitz, 1970, Shimer, 2008). Machin and Manning (1999) argue that the bulk of this duration dependence is accounted for by observed and unobserved heterogeneity, and not by "true" negative duration dependence. Moreover, Barron et al. (1985) and Holzer et al. (1991) find that vacancies that offer higher wages are filled more rapidly, as they attract more applicants. Learning from search can reconcile these observations on the two sides of the market. True positive unemployment duration dependence is an implication of the trade-off between wage and matching probability in directed search. As a worker becomes more pessimistic after match failure, he chooses to search for lower wages which come with higher matching probabilities. The flip side of this result is that the rate at which vacancies are filled increases with their wage offer. The reason why the average job-finding probability can decrease with unemployment duration is that the ability composition of workers in any given cohort deteriorates with unemployment duration. That is, as high-ability workers are matched more quickly than low-ability workers, the fraction of low-ability workers remaining in an unemployed cohort increases with unemployment duration. This deterioration of the ability composition reduces the average matching probability of the cohort.

More precisely, consider a cohort of unemployed workers with beliefs μ. For any given ability a_i, the job-finding probability $a_i g(\mu)$ increases as μ deteriorates in the course of unemployment. However, the average job-finding probability for these workers, given by (6.5), is a decreasing function of the ratio of low-ability workers to high-ability workers in the cohort, $\hat{u}_L(\mu)/\hat{u}_H(\mu)$. From (6.1)-(6.4), we can use induction to prove that this ratio increases as μ deteriorates with unemployment. It is possible that this composition effect dominates the effect of $g(\mu)$, in which case the average job-finding probability decreases with unemployment duration.

Confounding the above composition effect, workers who become unemployed at the same time can differ in their beliefs μ because their histories of past unemployment can differ. Since these histories are summarized by workers' pre-unemployment wages, as explained above, controlling for workers' pre-unemployment wages as well as the ability composition within each cohort and other characteristics of workers may be useful in empirical investigations of unemployment duration dependence.

(iii) *Desired wages versus reservation wages.* Influenced largely by the simplicity of McCall's (1970) search model, a large literature has been concerned with the determinants of the reservation wage, typically taking the arrival rate of wage offers as given. In particular, the potential sources of declining reservation wages have received much attention (see Burdett and Vishwanath, 1988, and the references therein). Learning from search also generates declining reservation wages. In contrast with partial equilibrium theories, our model links job-finding probabilities negatively to desired wages rather than reservation wages. The focus on desired wages is of practical relevance in empirical analyses of duration dependence. First, when search is directed, desired wages rather than reservation wages are the object that is tightly related to job-finding probabilities and unemployment durations. Evidence of the importance of directed search in the labor market is provided by Hall and Krueger (2008) and Holzer et al. (1991).[20] Second, a large empirical literature has found that variations in unemployment duration seem largely due to variations in the offer arrival rate and not in reservation wages (e.g., Devine and Kiefer, 1991). Third, in contrast to reservation wages, desired wages are observed once workers are reemployed. Even for the workers who fail to get a match, their desired wages can be inferred from their pre-unemployment wages and the current unemployment duration together.

(iv) *Wage dispersion and heterogeneity among non-employed workers.* A large literature documents wage dispersion among workers with similar observable characteristics (e.g, Mortensen, 2003). Motivated by this evidence, a theoretical literature explains wage dispersion among *ex ante* identical workers (e.g., Burdett and Judd, 1983, Burdett and Mortensen, 1998). Recently, Hornstein et al. (2007) have argued that conventional random matching models account for only a small fraction of wage dispersion among workers with similar characteristics. The problem lies in the difficulty to explain the coexistence of relatively short unemployment durations on average and relatively large measures of wage dispersion. Our theory suggests that endogenous heterogeneity among unemployed workers may explain part of the observed wage dispersion. Since unemployed workers can have different labor market histories and, hence, different beliefs about their ability to form a productive match, their reemployment wages can also differ.[21] This theory also provides

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[20] Using a survey of US workers, Hall and Krueger (2008) find that about 84% of all workers in the sample either know exactly or have a pretty good idea about how much a job would pay before the first interview for the job. The evidence in Holzer et al. (1991) on the positive relationship between the level of a wage offer and the number of applicants is also an indication of directed search.

[21] A complementary explanation is endogenous heterogeneity among employed workers that is generated by the combination of on-the-job search and wage-tenure contracts, as in the model by Shi (2009).

a natural explanation for the significant heterogeneity among non-employed workers in their attachment to the labor market and their transitions into employment that Jones and Riddell (1999) document.

(v) *The role of information in aggregate fluctuations.* Learning from search can interact with aggregate shocks in a non-trivial way. For example, consider an unexpected decrease in aggregate labor productivity, y. By itself, this negative shock has the conventional effects of reducing vacancies, wages and aggregate output. However, there is an additional effect through learning. The reduction in y is likely to reduce workers' matching probabilities for any given beliefs and, hence, makes search outcomes less informative. Thus, when an unemployed worker fails to find a job during a recession, the worker is relatively less sure about whether the failure is due to his low ability or to the depressed aggregate conditions. Consequently, all workers may search for a relatively longer time during recessions. Although this is speculative, it is indicative of the scope of the implications that learning from search can have for the analysis of labor market outcomes.

8. Conclusion

In this paper, we have proposed an equilibrium theory of learning from search in the labor market. The main assumption is that unemployed workers have incomplete information about their job-finding ability and, therefore, learn about their ability from their search outcomes. Success and failure of search both convey useful information about a worker's type. As workers experience different search outcomes, their labor market histories and, hence, their beliefs about their ability diverge. Firms cater to these workers by offering different wages. The theory formalizes a notion akin to discouragement. That is, over each unemployment spell, unemployed workers update their beliefs about their job-finding ability downward and reduce not only reservation wages, but also desired wages.

One contribution of the paper has been to integrate learning from search into an equilibrium framework. The equilibrium analysis is made tractable by modeling the search process as one in which terms of trade direct workers' search. With directed search, workers sort into different submarkets according to their beliefs of their ability, which severs the dependence of search behavior and market tightness on the wage distribution. This feature of block recursivity of the equilibrium enables us to examine jointly the workers' search behavior, the incentives to create jobs, and the wage distribution. By exploiting the fact that block recursivity is robust to the introduction of ex ante heterogeneity among workers/firms, our framework can be extended to include alternative sources of heterogeneity that would be important in empirical analyses of the labor market.

Another contribution of the paper has been to identify and explore a connection between convexity of a worker's value function in beliefs and the property of supermodularity. This connection enabled us to establish the properties of desired wages and optimal search behavior despite the potential presence of non-differentiable value functions and multiple solutions to a worker's optimization problem. This connection is likely to be useful in many other learning problems, because convexity of the value function in beliefs is inherent to optimal learning from experience.

The equilibrium theory of learning from search has provided a novel mechanism for generating endogenous heterogeneity among unemployed workers. The learning process turns *ex ante* identical workers into *ex post* heterogeneous workers who differ in posterior beliefs about their job-finding probabilities. We have argued that such endogenous heterogeneity provides a useful mechanism to understand a wide set of empirical observations about the labor market, including unemployment duration dependence and wage dispersion.

Appendix

A. Proof of Lemma 3.1

First, we prove existence of the equilibrium. The analysis leading to Lemma 3.1 has established the result that if the value function V obeys (3.8), then V and the optimal choices $G(\mu)$ exist. In section 6 we characterize the steady state distribution of workers. Thus, for existence of an equilibrium it suffices to show that Assumption 2 is sufficient for all matches to be accepted, in which case V indeed obeys (3.8).

Consider a worker with beliefs $\mu \in M$ who obtains a match in submarket $x \in X$, where $M = [a_L, a_H]$ and $X = [0, 1/a_H]$. If the worker accepts the match, the present value from the next period onward is $J_e(\phi(\mu), W(x))$. If the worker rejects the match, the value is $V(\phi(\mu))$. Thus, accepting a match is strictly preferred to rejecting a match if and only if $J_e(\phi(\mu), W(x)) > V(\phi(\mu))$ for all μ and x. Using (3.6), we can rewrite the latter condition as $W(x) > \frac{r+\sigma}{1+r} V(\phi(\mu))$. Since $W(x)$ is a decreasing function and $\phi(\mu) \leq a_H$, a sufficient condition for this requirement is

$$W(\frac{1}{a_H}) > \frac{r+\sigma}{1+r} V(a_H). \tag{A.1}$$

Part (1) of Appendix B derives $V(a_H)$ as in (B.1). Substituting (B.1), we rewrite (A.1) as

$$\frac{y-b}{c} > [A + a_H x_H]\, \lambda'(x^*) - a_H \lambda(x_H),$$

where x^* is defined by $\lambda'(x^*) = a_H \lambda(\frac{1}{a_H})$. The right side of the inequality is maximized at $x_H = x^*$. Thus, Assumption 2 is sufficient for (A.1) and so for all matches to be accepted.

Second, we prove that optimal choices are strictly positive, i.e., that $g(\mu) > 0$ for all $g(\mu) \in G(\mu)$ and all $\mu \in M$. Let $\mu \in M$ be an arbitrary level of beliefs and $g(\mu)$ an arbitrary selection from the set of optimal choices $G(\mu)$. Suppose that $g(\mu) = 0$, contrary to the lemma. In this case, the return to search is $R(0, \mu) = V(\mu)/(1+r)$, and the Bellman equation for V implies $V(\mu)/(1+r) = b/(r+\sigma)$. On the other hand, because the choice $x = 0$ is always feasible, we have: $V(\mu)/(1+r) \geq b/(r+\sigma)$. Substituting this lower bound on V, we have: $R(x, \mu) \geq \tilde{R}(x, \mu)$ for all x and μ, where

$$\tilde{R}(x, \mu) = \frac{x\mu}{A} \left[\frac{W(x)}{1-\sigma} + \frac{\delta b}{r+\sigma} \right] + (1 - x\mu) \frac{b}{r+\sigma}.$$

Thus, $\max_x R(x, \mu) \geq \max_x \tilde{R}(x, \mu)$. Note that $\tilde{R}(x, \mu)$ is differentiable and strictly concave in x. Substituting the function $W(.)$, we can verify that $\tilde{R}_1(0, \mu) > 0$ if and only if $(y-b)/c > A\lambda'(0)$. Since the latter condition is satisfied (see Remark 1), then $\max_x \tilde{R}(x, \mu) > \tilde{R}(0, \mu) = \frac{b}{r+\sigma}$. This result implies that $V(\mu) > b + (1-\sigma)\tilde{R}(0, \mu) = \frac{1+r}{r+\sigma} b$, which contradicts the supposition that $g(\mu) = 0$. Therefore, $g(\mu) > 0$ for all μ.

Third, (weak) convexity of V follows from standard arguments (e.g., Nyarko, 1994, Proposition 3.2). Because a convex function is almost everywhere differentiable (see Royden, 1988, pp113-114), V is almost everywhere differentiable.

Finally, we prove that V is strictly increasing. Let $TV(\mu)$ denote the right-hand side of (3.8) so that V is a fixed point of the mapping T. Let $C_1(M)$ be the set containing all bounded, continuous and increasing functions on M. Let $C_1^s(M)$ be the subset of $C_1(M)$ that contains all strictly increasing functions. We prove that $T : C_1(M) \to C_1^s(M)$. Once this is done, the argument of contraction mapping implies that $V \in C_1^s(M)$. To prove that $T : C_1(M) \to C_1^s(M)$, take any function $V \in C_1(M)$ and use it to calculate R and, hence, TV. We need to prove that $TV(\mu_a) > TV(\mu_b)$ for all $\mu_a, \mu_b \in M$ with $\mu_a > \mu_b$. Denote $g_i = g(\mu_i) \in G(\mu_i)$, where $i \in \{a, b\}$. We have:

$$\begin{array}{rl} & R(g_a,\mu_a) - R(g_b,\mu_b) \\ \geq & R(g_b,\mu_a) - R(g_b,\mu_b) \\ \geq & g_b(\mu_a-\mu_b)\left\{\frac{1}{A}\left[\frac{W(g_b)}{1-\sigma}+\delta\frac{V(\phi(\mu_b))}{1+r}\right]-\frac{V(H(g_b,\mu_b))}{1+r}\right\} \\ > & \frac{1}{1+r}g_b(\mu_a-\mu_b)\left[V(\phi(\mu_b))-V(H(g_b,\mu_b))\right] \geq 0. \end{array} \tag{A.2}$$

The first inequality comes from the fact that $g_i \in \arg\max_x R(x, \mu_i)$ and the second one from $V(H(g_b, \mu_a)) \geq V(H(g_b, \mu_b))$. The strict inequality uses the fact that $g_b > 0$ and that Assumption 2 implies $W(x) > \frac{r+\sigma}{1+r}V(\phi(\mu))$ for all x and μ (see above proof). The last inequality comes from $\phi(\mu_b) \geq H(g_b, \mu_b)$. Hence, $TV(\mu_a) > TV(\mu_b)$. **QED**

B. Optimal Choices at $\mu = a_H$ and $\mu = a_L$

In this appendix, we establish several useful results regarding optimal choices when beliefs lie at the two ends, a_H and a_L.

(1) The optimal search decision of a worker with beliefs $\mu = a_H$. Because $\phi(a_H) = H(x, a_H) = a_H$ for all x, then

$$R(x,a_H)=\frac{xa_H}{A}\left[\frac{W(x)}{1-\sigma}+\delta\frac{V(a_H)}{1+r}\right]+(1-xa_H)\frac{V(a_H)}{1+r},$$

and $V(a_H) = b + (1-\sigma)\max_x R(x, a_H)$. Condition (iii) in (3.12) implies that $R(x, a_H)$ is strictly concave in x, and so the optimal choice of x is unique under $\mu = a_H$. Let x_H be this optimal choice. Since $R(x, a_H)$ is also differentiable with respect to x, and since $x_H > 0$ by Lemma 3.1, x_H satisfies $R_1(x_H, a_H) \geq 0$, with equality if $x_H < 1/a_H$. Moreover, we can solve:

$$\frac{V(a_H)}{1+r}=\frac{Ab+a_Hx_HW(x_H)}{(r+\sigma)\left[A+a_Hx_H\right]}. \tag{B.1}$$

(2) The condition (4.2) is necessary and sufficient for $x_H < 1/a_H$. It is clear that $x_H < 1/a_H$ if and only if $R_1(1/a_H, a_H) < 0$. Computing the derivative $R_1(x, a_H)$, we conclude that $R_1(1/a_H, a_H) < 0$ if and only if

$$W(\frac{1}{a_H})+\frac{1}{a_H}W'(\frac{1}{a_H})<\frac{r+\sigma}{1+r}V(a_H). \tag{B.2}$$

Substituting $V(a_H)$ from (B.1) and $W(x)$ from (3.11), we find that (B.2) is equivalent to

$$\frac{y-b}{c} < [A + a_H x_H]\, \lambda'(\frac{1}{a_H}) - a_H \lambda(x_H).$$

The right-hand side of the above inequality is an increasing function of x_H, and its value at $x_H = 1/a_H$ is equal to the right-hand side of (4.2). Since $x_H \le 1/a_H$, (4.2) is necessary for the above condition and, hence, necessary for $x_H < 1/a_H$. On the other hand, if the optimal choice is $x_H = 1/a_H$, then $R_1(1/a_H, a_H) \ge 0$, and $V(a_H)$ is given by (B.1) with $x_H = 1/a_H$. Substituting this value of $V(a_H)$, we find that the condition $R_1(1/a_H, a_H) \ge 0$ violates (4.2). Thus, (4.2) is also sufficient for $x_H < 1/a_H$.

(3) Similar results at $\mu = a_L$. Similar to the above analysis, we can examine the optimal search decision of a worker with beliefs $\mu = a_L$, denoted as x_L. Because $\phi(a_L) = H(x, a_L) = a_L$ for all x, the analysis yields the results that $R(x, a_L)$ is strictly concave and differentiable in x and that $R_1(x_L, a_L) \ge 0$, with strict inequality only if $x_L = 1/a_H$. The value function satisfies:

$$\frac{V(a_L)}{1+r} = \frac{Ab + a_L x_L W(x_L)}{(r+\sigma)\,[A + a_L x_L]}. \tag{B.3}$$

Note that x_H and x_L both satisfy the condition $R_1(x_i, a_i) \ge 0$, where $i \in \{H, L\}$. By inspecting this condition and using strict monotonicity of the value function, we can deduce that $x_L \ge x_H$, where the inequality is strict if $x_H < 1/a_H$ (i.e., if (4.2) is satisfied).

(4) An upper bound on $V'(a_H^-)$ and a lower bound on $V'(a_L^+)$. These one-sided derivatives exist because V is continuous and convex (see Royden, 1988, pp113-114). Let $\varepsilon > 0$ be a sufficiently small number. For $V'(a_H^-)$, we can compute

$$\frac{V(a_H) - V(a_H - \varepsilon)}{1-\sigma} = R(x_H, a_H) - R(g(a_H - \varepsilon), a_H - \varepsilon) \le R(x_H, a_H) - R(x_H, a_H - \varepsilon).$$

Dividing this inequality by ε and taking the limit $\varepsilon \downarrow 0$, we obtain $V'(a_H^-) \le (1-\sigma) R_2(x_H, a_H^-)$. Note that $\phi(a_H) = a_H = H(x_H, a_H)$, $\phi'(a_H) = a_L/a_H$, and $H_2(x_H, a_H) = (1 - x_H a_L)/(1 - x_H a_H)$. Using these facts, we can compute:

$$R_2(x_H, a_H^-) = \frac{x_H}{(1-\sigma)A}\left[W(x_H) - \frac{r+\sigma}{1+r} V(a_H^-)\right] + \left[1 - \frac{(r+\sigma) x_H a_L}{(1-\sigma)A}\right] \frac{V'(a_H^-)}{1+r}.$$

Substituting into the inequality $V'(a_H^-) \le (1-\sigma) R_2(x_H, a_H^-)$, we get:

$$\frac{V'(a_H^-)}{1+r} \le \frac{x_H}{(r+\sigma)[A + x_H a_L]}\left[W(x_H) - \frac{r+\sigma}{1+r} V(a_H^-)\right].$$

Substituting (B.1) for $V(a_H)$, we can further write the above result as

$$\frac{V'(a_H^-)}{1+r} \le \frac{A x_H\,[W(x_H) - b]}{(r+\sigma)[A + x_H a_L]\,[A + a_H x_H]}.$$

Similarly, we can derive the following lower bound:

$$\frac{V'(a_L^+)}{1+r} \geq \frac{Ax_L\,[W(x_L)-b]}{(r+\sigma)[A+x_La_H]\,[A+a_Lx_L]}.$$

(5) The upper bound $\bar{\delta}$ that ensures $\delta/A < V'(a_L^+)/V'(a_H^-)$ for all $\delta \leq \bar{\delta}$. Substituting the above bounds on $V'(a_H^-)$ and $V'(a_L^+)$, we find that

$$\frac{V'(a_L^+)}{V'(a_H^-)} \geq \frac{x_L\,[W(x_L)-b]\,[A+x_Ha_L]\,[A+a_Hx_H]}{x_H\,[W(x_H)-b]\,[A+x_La_H]\,[A+a_Lx_L]}.$$

Recall that for $i \in \{L,H\}$, the optimal choice x_i satisfies $R_1(x_i,a_i) \geq 0$ and the value function satisfies $V(a_i) > \frac{1+r}{r+\sigma}b$. Using these results, we can verify that the function $x[W(x)-b]$ is strictly increasing in x for $x \in [x_H, x_L]$. Because $x_L \geq x_H$ (see the proof above), and so $x_L\,[W(x_L)-b] \geq x_H[W(x_H)-b]$. Substituting this result and the facts that $x_H > 0$ and $x_L \leq 1/a_H$, we conclude that

$$\frac{V'(a_L^+)}{V'(a_H^-)} > \frac{A^2}{[A+1]\left[A+\frac{a_L}{a_H}\right]}.$$

Substituting this bound, we find that a sufficient condition for the requirement $\delta/A < V'(a_L^+)/V'(a_H^-)$ is $\Omega(\delta) \geq 0$, where

$$\Omega(\delta) = \frac{r+\sigma}{1-\sigma}\left(\frac{r+\sigma}{1-\sigma}+\delta\right)^2 - \delta\left[\left(1+\frac{a_L}{a_H}\right)\left(\frac{r+\sigma}{1-\sigma}+\delta\right)+\frac{a_L}{a_H}\right].$$

The function $\Omega(\delta)$ is quadratic and it involves only the parameters of the model. Because $\Omega(0) > 0$, there exists $\bar{\delta} > 0$ such that $\Omega(\delta) \geq 0$ for all $\delta \in [0,\bar{\delta}]$. Thus, $\delta/A < V'(a_L^+)/V'(a_H^-)$ for all $\delta \in [0,\bar{\delta}]$.

C. Proof of Theorem 4.1

Take arbitrary z_a, $z_b \in -X$ and arbitrary μ_a, $\mu_b \in M$, with $z_a > z_b$ and $\mu_a > \mu_b$. Denote:

$$D = \left[\hat{R}(z_a,\mu_a) - \hat{R}(z_a,\mu_b)\right] - \left[\hat{R}(z_b,\mu_a) - \hat{R}(z_b,\mu_b)\right].$$

We need to show $D > 0$. Temporarily denote $\phi_j = \phi(\mu_j)$, $H_{ij} = H(-z_i,\mu_j)$ and $V_{ij} = V(H_{ij})$, where $i,j \in \{a,b\}$. Computing D, we have:

$$(1+r)D = D_1 - \frac{\delta}{A}[V(\phi_a)-V(\phi_b)](z_a-z_b),$$

where

$$D_1 = (z_a+\frac{1}{\mu_a})V_{aa} - (z_b+\frac{1}{\mu_a})V_{ba} - (z_a+\frac{1}{\mu_b})V_{ab} + (z_b+\frac{1}{\mu_b})V_{bb}.$$

Denote $\tilde{H} = \min\{H_{ba}, H_{ab}\}$. Because $H(-z,\mu)$ is a strictly increasing function of z and μ for all $\mu \in (a_L, a_H)$, then $H_{aa} > \tilde{H} \geq H_{bb}$. Because V is convex, we have:

$$\min\left\{\frac{V_{aa}-V_{ba}}{H_{aa}-H_{ba}}, \frac{V_{aa}-V_{ab}}{H_{aa}-H_{ab}}\right\} \geq \frac{V_{aa}-V(\tilde{H})}{H_{aa}-\tilde{H}} \geq \frac{V_{aa}-V_{bb}}{H_{aa}-H_{bb}}.$$

Substituting V_{ba}, V_{ab} and V_{bb} from these inequalities, and substituting H, we have:

$$\begin{aligned} D_1 &\geq \tfrac{V_{aa}-V(\tilde{H})}{H_{aa}-\tilde{H}} \Big\{ \tfrac{1}{\mu_a}\left[(1+z_a\mu_a)H_{aa} - (1+z_b\mu_a)H_{ba}\right] \\ &\qquad\qquad -\tfrac{1}{\mu_b}\left[(1+z_a\mu_b)H_{ab} - (1+z_b\mu_b)H_{bb}\right]\Big\} \\ &= \tfrac{V_{aa}-V(\tilde{H})}{H_{aa}-\tilde{H}}(z_a - z_b)(\phi_a - \phi_b). \end{aligned}$$

Thus, $D > 0$ if

$$\frac{\delta}{A} < \left[\frac{V_{aa}-V(\tilde{H})}{H_{aa}-\tilde{H}}\right] \Big/ \left[\frac{V(\phi_a)-V(\phi_b)}{\phi_a-\phi_b}\right].$$

Because V is convex, then

$$\frac{V_{aa}-V(\tilde{H})}{H_{aa}-\tilde{H}} \geq V'(a_L^+); \quad \frac{V(\phi_a)-V(\phi_b)}{\phi_a-\phi_b} \leq V'(a_H^-),$$

where $V'(\mu^+)$ is the right derivative, and $V'(\mu^-)$ the left derivative, of V at μ. Hence, a sufficient condition for $D > 0$ is $\delta/A < V'(a_L^+)/V'(a_H^-)$, which is implied by Assumption 3.

Thus, the function $\hat{R}(z,\mu)$ is strictly supermodular. Because $-X$ is a lattice, the monotone selection theorem in Topkis (1998, Theorem 2.8.4, p79) implies that every selection from $Z(\mu)$ is increasing. As a result, every selection $g(\mu)$ from $G(\mu)$ is decreasing, and $w(\mu) = W(g(\mu))$ is increasing. **QED**

D. Proof of Theorem 4.2

Before proving Theorem 4.2, it is useful to establish first the following lemma:

Lemma D.1. *For any given z, the functions $\mu V(\phi(\mu))$ and $(1+z\mu)V(H(-z,\mu))$ are convex in μ if $V(.)$ is convex, and strictly convex in μ if $V(.)$ is strictly convex.*

Proof. Assume that V is convex, we first prove that $\mu V(\phi(\mu))$ is convex in μ. Take two arbitrary levels, μ_a, $\mu_b \in M$, with $\mu_a > \mu_b$. Let γ be an arbitrary number in $(0,1)$, and define $\mu_\gamma = \gamma\mu_a + (1-\gamma)\mu_b$. We need to prove that $\mu_\gamma V(\phi(\mu_\gamma)) \leq \gamma\mu_a V(\phi(\mu_a)) + (1-\gamma)\mu_b V(\phi(\mu_b))$. Shorten the notation $\phi(\mu_i)$ to ϕ_i, where $i \in \{a, b, \gamma\}$. Denote $\kappa = (\phi_\gamma - \phi_b)/(\phi_a - \phi_b)$. Clearly, $\kappa \in [0,1]$, and $\phi_\gamma = \kappa\phi_a + (1-\kappa)\phi_b$. Moreover, since $\mu\phi(\mu)$ is a linear function of μ, we can verify that $\kappa\mu_\gamma = \gamma\mu_a$ and $(1-\kappa)\mu_\gamma = (1-\gamma)\mu_b$. Thus,

$$\begin{aligned} &\mu_\gamma V(\phi_\gamma) = \mu_\gamma V(\kappa\phi_a + (1-\kappa)\phi_b) \\ &\leq \mu_\gamma\left[\kappa V(\phi_a) + (1-\kappa)V(\phi_b)\right] = \gamma\mu_a V(\phi_a) + (1-\gamma)\mu_b V(\phi_b). \end{aligned}$$

The inequality comes from convexity of V and the fact that $\mu_\gamma > 0$. The last equality comes from the facts that $\kappa\mu_\gamma = \gamma\mu_a$ and $(1-\kappa)\mu_\gamma = (1-\gamma)\mu_b$. Thus, $\mu V(\phi(\mu))$ is convex in μ if V is convex. If V is strictly convex, then the above inequality is strict, in which case $\mu V(\phi(\mu))$ is strictly convex.

Note that the function $(1+z\mu)H(-z,\mu)$ is also linear in μ for any given z. Thus, applying a similar proof as the above establishes the convexity properties of the function $(1+z\mu)V(H(-z,\mu))$. **QED**

We now prove Theorem 4.2 by establishing the following relationships among the five statements in the theorem.

(i) $\Longleftrightarrow$ (ii): Optimal learning has the following standard property (see Nyarko, 1994, Proposition 4.1): The value function is strictly convex in beliefs if and only if there do not exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice z_0 such that $z_0 \in Z(\mu)$ for all $\mu \in [\mu_b, \mu_a]$. Since $z(\mu)$ is an increasing function by Theorem 4.1, the standard property implies that V is strictly convex if and only if every selection $z(\mu)$ is strictly increasing for all μ.

(ii) $\Longrightarrow$ (iii): Suppose $\{-1/a_H\} \in Z(\mu_a)$ for some $\mu_a > a_L$ so that (iii) is violated. Because every selection $z(\mu)$ is increasing, then $Z(\mu)$ contains only the singleton $\{-1/a_H\}$ for all $\mu < \mu_a$. In this case, (ii) does not hold for $\mu \le \mu_a$.

(iii) $\Longrightarrow$ (iv): This follows from $a_H > a_L$.

(iv) $\Longleftrightarrow$ (v): See part (2) of Appendix B.

(iv) $\Longrightarrow$ (i): We show that a violation of (i) implies that $\{-1/a_H\} \in Z(a_H)$, which violates (iv). Suppose that V is not strictly convex. Proposition 4.1 in Nyarko (1994) implies that there exist μ_a and μ_b in M, with $\mu_a > \mu_b$, and a choice z_0 such that $z_0 \in Z(\mu)$ for all $\mu \in [\mu_b, \mu_a]$. Since $\mu_a > \mu_b$, there exist μ_a' and μ_b', with $\mu_a > \mu_a' > \mu_b' > \mu_b$, such that the supposition also holds for $\mu \in [\mu_b', \mu_a']$. Thus, without loss of generality, we assume that $\mu_b > a_L$ and $\mu_a < a_H$. In this case, Proposition 4.1 in Nyarko (1994) shows that $V(\mu)$ is linear for all $\mu \in [\mu_b, \mu_a]$. In turn, this result implies that $V(\mu)$ must be linear for all $\mu \in [\phi(\mu_b), \phi(\mu_a)]$: If $V(\mu)$ were strictly convex in any subinterval of $[\phi(\mu_b), \phi(\mu_a)]$, Lemma D.1 above would imply that $R(-z_0, \mu)$ is strictly convex in some subinterval of $[\mu_b, \mu_a]$. Similarly, $V(\mu)$ must be linear for all $\mu \in [H_b, H_a]$, where H_i denotes $H(-z_0, \mu_i)$ for $i \in \{a, b\}$. Let $V'(\phi_b)$ denote the slope of the linear section of V for $\mu \in [\phi(\mu_b), \phi(\mu_a)]$ and $V'(H_b)$ the slope of the linear section for $\mu \in [H_b, H_b]$. For all $\mu \in [\mu_b, \mu_a]$, we have

$$\begin{aligned} R(-z,\mu) = \ & -z\mu\tfrac{W(-z)}{(1-\sigma)A} - \tfrac{\delta z\mu}{(1+r)A}\{V(\phi(\mu_b)) + V'(\phi_b)[\phi(\mu) - \phi(\mu_b)]\} \\ & + \tfrac{1+z\mu}{1+r}\{V(H_b) + V'(H_b)[H(-z,\mu) - H_b]\}. \end{aligned}$$

Because $\mu\phi(\mu)$ and $(1+z\mu)H(-z,\mu)$ are linear in μ for any given z, the last two terms in the above expression are linear in z. In this case, part (iii) in (3.12) implies that $R(-z,\mu)$ is strictly concave and differentiable in z for all $\mu \in [\mu_b, \mu_a]$, and so the optimal choice $z(\mu)$ is unique. By the supposition, this optimal choice is $z(\mu) = z_0$ for all $\mu \in [\mu_b, \mu_a]$. Using these results and the fact that $z_0 < 0$ (see Lemma 3.1), we conclude that that z_0 satisfies the complementary slackness condition, $R_1(-z_0, \mu) \ge 0$ and $z_0 \ge -1/a_H$, where R_1 denotes the derivative of R with respect to $-z$.

If $z_0 > -1/a_H$, then $R_1(-z_0, \mu) = 0$. In this case, because $R_1(-z, \mu)$ is differentiable with respect to z and μ for all $\mu \in [\mu_b, \mu_a]$, we can compute $dz_0/d\mu = R_{12}(-z_0, \mu)/R_{11}(-z_0, \mu)$. Thus, $dz_0/d\mu > 0$ if and only if $R_{12}(-z_0, \mu) < 0$. The latter condition is equivalent to $\delta/A < V'(H_b)/V'(\phi_b)$ after using the first-order condition, $R_1(-z_0, \mu) = 0$. Because $\phi_b < a_H$ and $H_b \geq a_L$, convexity of V implies $V'(\phi_b) \leq V'(a_H^-)$ and $V'(H_b) \geq V'(a_L^+)$. Thus, a sufficient condition for $R_{12}(-z_0, \mu) < 0$ is that $\delta/A < V'(a_L^+)/V'(a_H^-)$, which in turn is implied by Assumption 3 (see part (5) of Appendix B). Hence, Assumption 3 implies that if $z_0 > -1/a_H$, then $dz_0/d\mu > 0$. Because this result contradicts the supposition that z_0 is constant for all $\mu \in [\mu_b, \mu_a]$, the unique optimal choice must be $z_0 = -1/a_H$.

Next, consider the interval of $\mu \in [\phi(\mu_b), \phi(\mu_a)]$. Since V is linear over this interval, there is a choice z_1 that is optimal for all μ in this interval. Repeating the above argument for this interval, we conclude that V is linear for $\mu \in [\phi^2(\mu_b), \phi^2(\mu_a)]$, where $\phi^2(\mu) = \phi(\phi(\mu))$. Moreover, the optimal choice is uniquely given as $z_1 = -1/a_H$. We can repeat the argument for all $\mu \in [\phi^i(\mu_b), \phi^i(\mu_a)]$, where $\phi^i(\mu) = \phi(\phi^{i-1}(\mu))$ and $i = 1, 2, ...$ That is, V is linear for $\mu \in [\phi^i(\mu_b), \phi^i(\mu_a)]$, in which the optimal choice is uniquely given as $-1/a_H$.

The function $\phi(\mu)$ is defined as $\phi(\mu) = a_H + a_L - \frac{a_H a_L}{\mu}$. Clearly, $\phi(a_H) = a_H$, $\phi(a_L) = a_L$, and $\phi(\mu) > \mu$ for all $\mu \in (a_L, a_H)$. With these properties of ϕ, we know that for any arbitrary $\mu_c \in (a_L, a_H)$, the sequence $\{\phi^i(\mu_c)\}_{i=0}^{\infty}$ converges to a_H. The argument in the previous paragraph implies that for every positive integer i, the set of maximizers, $Z(\phi^i(\mu_c))$, contains only the singleton $\{-1/a_H\}$. Thus, $\lim_{i\to\infty} z(\phi^i(\mu_c)) = -1/a_H$. Because Z is upper hemi-continuous, we conclude that $\{-1/a_H\} \in Z(a_H)$. **QED**

E. Proof of Lemma 5.1

First, Theorem 4.2 implies that under (4.2), the optimal choice $z(\mu)$ is interior for all $\mu > a_L$. Next, we show that $V'(h(\mu))$ exists for all $\mu \in (a_L, a_H)$, where $h(\mu) = H(-z(\mu), \mu)$. For any real number r, define $r^- = \lim_{\varepsilon\downarrow 0}(r - \varepsilon)$ and $r^+ = \lim_{\varepsilon\downarrow 0}(r + \varepsilon)$. Fix $\mu \in (a_L, a_H)$. Optimality requires $\hat{R}_1(z^-(\mu), \mu) \geq \hat{R}_1(z^+(\mu), \mu)$. Note that a continuous, convex function has left and right derivatives. Because $W(-z)$ is continuous, V is continuous and convex, and H is continuously differentiable, then

$$\begin{aligned}\hat{R}_1(z^+(\mu), \mu) = \;& \tfrac{1}{A}\left[\tfrac{z(\mu)W'(-z(\mu)) - W(-z(\mu))}{1-\sigma} - \delta\tfrac{V(\phi(\mu))}{1+r}\right] \\ & + \tfrac{V(h(\mu))}{1+r} - \left(\tfrac{1}{\mu} + z(\mu)\right)\tfrac{V'(h^+(\mu))}{1+r} H_1(-z(\mu), \mu),\end{aligned}$$

$$\begin{aligned}\hat{R}_1(z^-(\mu), \mu) = \;& \tfrac{1}{A}\left[\tfrac{z(\mu)W'(-z(\mu)) - W(-z(\mu))}{1-\sigma} - \delta\tfrac{V(\phi(\mu))}{1+r}\right] \\ & + \tfrac{V(h(\mu))}{1+r} - \left(\tfrac{1}{\mu} + z(\mu)\right)\tfrac{V'(h^-(\mu))}{1+r} H_1(-z(\mu), \mu).\end{aligned}$$

Here we have used the results that $H(-z^+(\mu), \mu) = h^+(\mu)$ and $H(-z^-(\mu), \mu) = h^-(\mu)$, which come from the fact that $H(-z, \mu)$ is an increasing function of z. Recall that H_1 denotes the derivative of $H(-z, \mu)$ with respect to the first argument, rather than to z. Since $H_1 < 0$ and V is convex, the above expressions imply $\hat{R}_1(z^+(\mu), \mu) \geq \hat{R}_1(z^-(\mu), \mu)$.

However, the optimality of $z(\mu)$ requires $\hat{R}_1(z^+(\mu),\mu) \leq \hat{R}_1(z^-(\mu),\mu)$. It must be true that $\hat{R}_1(z^-(\mu),\mu) = \hat{R}_1(z^+(\mu),\mu)$, which implies

$$V'(h^-(\mu)) = V'(h^+(\mu)) = V'(h(\mu)).$$

Thus, optimal choices in every period satisfy the first-order condition. **QED**

F. Proofs of Lemma 5.2 and Theorem 5.3

First, we prove the following lemma (which does require optimal choices to be interior):

Lemma F.1. *$\bar{z}(\mu)$ is right-continuous and $\underline{z}(\mu)$ is left-continuous at each $\mu \in M$.*

Proof. Pick an arbitrary $\mu \in M$. Let $\{\mu_i\}$ be a sequence with $\mu_i \to \mu$ and $\mu_i \geq \mu_{i+1} \geq \mu$ for all i. Because $\bar{z}(\mu)$ is an increasing function, then $\{\bar{z}(\mu_i)\}$ is a decreasing sequence and $\bar{z}(\mu_i) \geq \bar{z}(\mu)$ for all i. Thus, $\bar{z}(\mu_i) \downarrow z_c$ for some $z_c \geq \bar{z}(\mu)$. On the other hand, the Theorem of the Maximum implies that the correspondence $Z(\mu)$ is upper hemi-continuous (uhc) (see Stokey and Lucas with Prescott, 1989, p62). Because $\mu_i \to \mu$, and $\bar{z}(\mu_i) \in Z(\mu_i)$ for each i, uhc of Z implies that there is a subsequence of $\{\bar{z}(\mu_i)\}$ that converges to an element in $Z(\mu)$. This element must be z_c, because all convergent subsequences of a convergent sequence must have the same limit. Thus, $z_c \in Z(\mu)$, and so $z_c \leq \max Z(\mu) = \bar{z}(\mu)$. Therefore, $\bar{z}(\mu_i) \downarrow z_c = \bar{z}(\mu)$, which shows that $\bar{z}(\mu)$ is right-continuous.

Similarly, by examining the sequence $\{\mu_i\}$ with $\mu_i \to \mu$ and $\mu \geq \mu_{i+1} \geq \mu_i$ for all i, we can show that $\underline{z}$ is left-continuous. This completes the proof of Lemma F.1.

Next, we prove Lemma 5.2. Fix $\mu_a \in (a_L, a_H)$. Because $\bar{z}(\mu)$ maximizes $R(-z,\mu)$ for each given μ, then

$$\begin{aligned} &V(\mu) = b + (1-\sigma)R(-\bar{z}(\mu),\mu) \geq b + (1-\sigma)R(-\bar{z}(\mu_a),\mu) \\ &V(\mu_a) = b + (1-\sigma)R(-\bar{z}(\mu_a),\mu_a) \geq b + (1-\sigma)R(-\bar{z}(\mu),\mu_a). \end{aligned}$$

Taking $\mu > \mu_a$, where $\mu_a < a_H$, and dividing the above inequalities by $(\mu - \mu_a)$, we obtain:

$$\frac{R(-\bar{z}(\mu_a),\mu) - R(-\bar{z}(\mu_a),\mu_a)}{\mu - \mu_a} \leq \frac{V(\mu) - V(\mu_a)}{(1-\sigma)(\mu - \mu_a)} \leq \frac{R(-\bar{z}(\mu),\mu) - R(-\bar{z}(\mu),\mu_a)}{\mu - \mu_a}.$$

Take the limit $\mu \downarrow \mu_a$. Under (4.2), $V'(H(-\bar{z}(\mu_a),\mu_a))$ exists for each μ (see Lemma 5.1). Because $\bar{z}(\mu)$ is right-continuous, $\lim_{\mu\downarrow\mu_a} \bar{z}(\mu) = \bar{z}(\mu_a)$. Thus, the first and last two ratios above both converge to the same limit, $R_2(-\bar{z}(\mu_a),\mu_a^+)$, where

$$\begin{aligned} & R_2(-\bar{z}(\mu_a),\mu_a^+) \\ = \; & \bar{z}(\mu_a)\left[-\tfrac{W(-\bar{z}(\mu_a))}{(1-\sigma)A} - \delta\tfrac{V(\phi(\mu_a))}{A(1+r)} + \tfrac{V(H(-\bar{z}(\mu_a),\mu_a))}{1+r}\right] \\ & -\tfrac{\mu_a\bar{z}(\mu_a)\delta}{A(1+r)}V'(\phi^+(\mu_a))\phi'(\mu_a) + [\mu_a\bar{z}(\mu_a)+1]\,\tfrac{V'(H(-\bar{z}(\mu_a),\mu_a))}{1+r}H_2(-\bar{z}(\mu_a),\mu_a). \end{aligned}$$

Thus, $V'(\mu_a^+) = (1-\sigma)R_2(-\bar{z}(\mu_a),\mu_a^+)$.

Now conduct the above exercise with $\underline{z}$ replacing $\bar{z}$. For $\mu < \mu_a$ and $\mu_a > a_L$, we have:

$$\frac{R(-\underline{z}(\mu_a), \mu) - R(-\underline{z}(\mu_a), \mu_a)}{\mu - \mu_a} \geq \frac{V(\mu) - V(\mu_a)}{(1-\sigma)(\mu - \mu_a)} \geq \frac{R\left(-\underline{z}(\mu), \mu\right) - R(-\underline{z}(\mu), \mu_a)}{\mu - \mu_a}.$$

Take the limit $\mu \uparrow \mu_a$. Because $\underline{z}(\mu)$ is left-continuous and interior, then the first and the last ratios above both converge to the same limit, $R_2(-\underline{z}(\mu_a), \mu_a^-)$, where

$$\begin{aligned} & R_2(-\underline{z}(\mu_a), \mu_a^-) \\ = \; & \underline{z}(\mu_a)\left[-\tfrac{W(-\underline{z}(\mu_a))}{(1-\sigma)A} - \delta\tfrac{V(\phi(\mu_a))}{A(1+r)} + \tfrac{V(H(-\underline{z}(\mu_a),\mu_a))}{1+r}\right] \\ & -\tfrac{\mu_a \underline{z}(\mu_a)\delta}{A(1+r)} V'(\phi^-(\mu_a))\phi'(\mu_a) + \left[\mu_a \underline{z}(\mu_a) + 1\right] \tfrac{V'(H(-\underline{z}(\mu_a),\mu_a))}{1+r} H_2(-\underline{z}(\mu_a), \mu_a). \end{aligned}$$

Thus, $V'(\mu_a^-) = (1-\sigma)R_2(-\underline{z}(\mu_a), \mu_a^-)$.

It is clear that $V'(\mu_a^+) \geq V'(\mu_a^-)$, because V is convex. To find the conditions for V to be differentiable at μ_a, use the definition $R(-z, \mu) = \mu \hat{R}(z, \mu)$ to compute:

$$R_2(-z(\mu), \mu) = \hat{R}(z(\mu), \mu) + \mu \hat{R}_2(z(\mu), \mu).$$

Note the following features. First, because $\hat{R}(z, \mu)$ is strictly supermodular, $\hat{R}_2(\bar{z}(\mu_a), \mu_a) \geq \hat{R}_2(\underline{z}(\mu_a), \mu_a)$, where the inequality is strict if and only if $\bar{z}(\mu_a) > \underline{z}(\mu_a)$. Second, μ_a^+ appears in the expression for $R_2(-z(\mu_a), \mu_a^+)$ only through the term $V'(\phi^+(\mu_a))$, and μ_a^- appears in the expression for $R_2(-z(\mu_a), \mu_a^-)$ only through the term $V'(\phi^-(\mu_a))$. Since V is strictly convex (and $\underline{z}, \bar{z} < 0$), we have $R_2(-z(\mu_a), \mu_a^+) \geq R_2(-z(\mu_a), \mu_a^-)$ for all $z(\mu_a) \in Z(\mu_a)$, where the inequality is strict if and only if $V'(\phi^+(\mu_a)) > V'(\phi^-(\mu_a))$. Third, $\hat{R}(\bar{z}(\mu), \mu) = \hat{R}(\underline{z}(\mu), \mu)$, since both $\bar{z}(\mu)$ and $\underline{z}(\mu)$ maximize $\hat{R}(z, \mu)$. These features imply:

$$\begin{aligned} R_2(-\bar{z}(\mu_a), \mu_a^+) & \geq R_2(-\bar{z}(\mu_a), \mu_a^-) \\ & \geq \hat{R}(\underline{z}(\mu_a), \mu_a^-) + \mu_a \hat{R}_2(\underline{z}(\mu_a), \mu_a^-) = R_2(-\underline{z}(\mu_a), \mu_a^-). \end{aligned}$$

The first inequality is strict if and only if V is not differentiable at $\phi(\mu_a)$, and the second inequality is strict if and only if $\bar{z}(\mu_a) > \underline{z}(\mu_a)$. Therefore, V is differentiable at μ_a if and only if V is differentiable at $\phi(\mu_a)$ and $\bar{z}(\mu_a) = \underline{z}(\mu_a)$. Finally, Lemma 5.1 states that for all $\mu \in M$, V is differentiable at $h(\mu)$. Thus, the optimal choice is unique and the value function is differentiable at all nodes on the tree $\Upsilon(h(\mu))$ for all $\mu \in \Upsilon(\mu_0)$. This completes the proof of Lemma 5.2.

Finally, we prove Theorem 5.3. Consider arbitrary initial beliefs $\mu_0 \in (a_L, a_H)$ and let $\Upsilon(\mu_0)$ be the tree of beliefs generated from μ_0 by the equilibrium. Suppose that V is differentiable at some $\mu \in \Upsilon(\mu_0)$. By Lemma 5.1, V is differentiable at $h(\mu)$ and, by Lemma 5.2, the optimal choice $z(\mu)$ is unique and V is differentiable at $\phi(\mu)$. Thus, V is differentiable at every node immediately following μ. Repeating this argument for each of these subsequent nodes, we conclude that the optimal choice is unique and the value function is differentiable at all nodes on the tree generated from μ by the equilibrium, i.e., at all $\mu' \in \Upsilon(\mu)$. **QED**

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